A Proofs

A.1 Bargaining and portfolio problems

The second-subperiod value functions can be written as

$$W_t^I(\boldsymbol{a}_t, a_t^b, k_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I \tag{128}$$

$$W_t^B(\boldsymbol{a}_t, a_t^b, k_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + a_t^b + k_t + \bar{W}_t^B \tag{129}$$

$$W_t^E(\boldsymbol{a}_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + \bar{W}_t^E, \tag{130}$$

where

$$\bar{W}_t^I \equiv T_t + \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \Big[-\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t \int V_{t+1}^I \left[\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s + (1-\eta) A^s, \varepsilon \right] dG(\varepsilon) \Big] \tag{131}$$

$$\bar{W}_t^B \equiv \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left[-\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t V_{t+1}^B (\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s) \right] \tag{132}$$

$$\bar{W}_t^E \equiv \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left[-\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t V_{t+1}^E (\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s) \right]. \tag{133}$$

Proof of Lemma 1. In a nonmonetary economy, (128) reduces to

$$W_t^I(\boldsymbol{a}_t, a_t^b, k_t) = \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t^I.$$

$(i)(a)$. In a nonmonetary equilibrium (3) implies $\overline{a}_{10t}^s(a_t^s, \varepsilon) = \arg\max_{0 \le \overline{a}_t^s \le a_t^s} (\varepsilon y_t + \phi_t^s) \overline{a}_t^s$.

$(i)(b)$. In a nonmonetary economy, (4) implies

$$\left[\overline{a}_{01t}^b(a_t^s, \varepsilon), k_{01t}(a_t^s, \varepsilon) \right] = \arg \max_{-\lambda \phi_t^s a_t^s \le \overline{a}_t^b \le 0, 0 \le k_t} \left(\overline{a}_t^b - k_t \right)^\theta k_t^{1-\theta}.$$

$(i)(c)$. In a nonmonetary economy, (5) implies $\left[\overline{a}_{11t}^s(a_t^s, \varepsilon), \overline{a}_{11t}^b(a_t^s, \varepsilon), k_{11t}(a_t^s, \varepsilon) \right]$ is the solution to

$$\max_{(\overline{a}_t^s, k_t) \in \mathbb{R}_+^2, \overline{a}_t^b \in \mathbb{R}} \left[(\varepsilon y_t + \phi_t^s)(\overline{a}_t^s - a_t^s) + \overline{a}_t^b - k_t \right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{\phi}_t^s \overline{a}_t^s + \overline{a}_t^b = \bar{\phi}_t^s a_t^s \tag{134}$$

$$-\lambda \phi_t^s \overline{a}_t^s \le \overline{a}_t^b. \tag{135}$$

Notice that the first-order condition with respect to k_t implies

$$k_{11t}(a_t^s, \varepsilon) = (1-\theta) \left\{ (\varepsilon y_t + \phi_t^s) \left[\overline{a}_{11t}^s(a_t^s, \varepsilon) - a_t^s \right] + \overline{a}_{11t}^b(a_t^s, \varepsilon) \right\}, \tag{136}$$

so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{\overline{a}_t^s \in \mathbb{R}_+, \overline{a}_t^b \in \mathbb{R}} \left[(\varepsilon y_t + \phi_t^s) (\overline{a}_t^s - a_t^s) + \overline{a}_t^b \right] \text{ s.t. (134), and (135).}$$

Since (134) implies $\overline{a}_t^b = \bar{\phi}_t^s (a_t^s - \overline{a}_t^s)$,

$$\overline{a}_{11t}^s (a_t^s, \varepsilon) = \arg\max_{\overline{a}_t^s} (\varepsilon - \varepsilon_t^n) \overline{a}_t^s \text{ s.t. } 0 \leq \overline{a}_t^s \text{ and } \left(\bar{\phi}_t^s - \lambda \phi_t^s \right) \overline{a}_t^s \leq \bar{\phi}_t^s a_t^s.$$

The problem has no solution (for $\varepsilon > \varepsilon_t^n$) if $\bar{\phi}_t^s - \lambda \phi_t^s \leq 0$. Provided $\bar{\phi}_t^s - \lambda \phi_t^s > 0$, the solution exists for all ε and is given by (15). Given $\overline{a}_{11t}^s (a_t^s, \varepsilon)$, $\overline{a}_{11t}^b (a_t^s, \varepsilon) = \bar{\phi}_t^s [a_t^s - \overline{a}_{11t}^s (a_t^s, \varepsilon)]$ as in (16), and $k_{11t} (a_t^s, \varepsilon)$ is given by (136), or equivalently, (17).

(*ii*) In a nonmonetary equilibrium, a bond broker's problem in the OTC is identical to that of an investor who is able to contact only the bond market.

(*iii*) In a nonmonetary equilibrium, an equity broker's problem in the OTC is identical to that of an investor who is able to contact only the equity market. ■

Proof of Lemma 2.

$(i)(a)$. With (128), it is easy to show that the solution to the optimization problem in (3) given by (21) and (22).

$(i)(b)$. With (128), (4) can be written as

$$\max_{(\overline{a}_t^m, k_t) \in \mathbb{R}_+^2, \overline{a}_t^b \in \mathbb{R}} \left[\phi_t^m \overline{a}_t^m + \overline{a}_t^b - k_t - \phi_t^m a_t^m \right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } q_t \overline{a}_t^b = a_t^m - \overline{a}_t^m$$
$$-\lambda \phi_t^s a_t^s \leq \overline{a}_t^b.$$

Notice that the first-order condition with respect to k_t implies $k_t = (1 - \theta) \left[\phi_t^m (\overline{a}_t^m - a_t^m) + \overline{a}_t^b \right]$, so the $\overline{a}_{01t}^m (\boldsymbol{a}_t, \varepsilon)$ and $\overline{a}_{01t}^b (\boldsymbol{a}_t, \varepsilon)$ can be found by solving the following auxiliary problem

$$\max_{\overline{a}_t^m \in \mathbb{R}_+, \overline{a}_t^b \in \mathbb{R}} \left[\phi_t^m \overline{a}_t^m + \overline{a}_t^b \right] \text{ s.t. } q_t \overline{a}_t^b = a_t^m - \overline{a}_t^m \text{ and } -\lambda \phi_t^s a_t^s \leq \overline{a}_t^b \tag{137}$$

and given $[\overline{a}_{01t}^m (\boldsymbol{a}_t, \varepsilon), \overline{a}_{01t}^b (\boldsymbol{a}_t, \varepsilon)]$, the fee is

$$k_{01t} (\boldsymbol{a}_t, \varepsilon) = (1 - \theta) \left\{ \phi_t^m [\overline{a}_{01t}^m (\boldsymbol{a}_t, \varepsilon) - a_t^m] + \overline{a}_{01t}^b (\boldsymbol{a}_t, \varepsilon) \right\}. \tag{138}$$

Thus the solution to (4) is given by (23) and (25), and (26).

$(i)(c)$. With (128), (5) can be written as

$$\max_{(\overline{a}_t^m, \overline{a}_t^s, k_t) \in \mathbb{R}_+^3, \overline{a}_t^b \in \mathbb{R}} \left\{ (\varepsilon y_t + \phi_t^s) \left[\overline{a}_t^s - \overline{a}_{10t}^s(\boldsymbol{a}_t, \varepsilon) \right] + \phi_t^m \left[\overline{a}_t^m - \overline{a}_{10t}^m(\boldsymbol{a}_t, \varepsilon) \right] + \overline{a}_t^b - k_t \right\}^\theta k_t^{1-\theta}$$

$$\text{s.t. } \overline{a}_t^m + p_t \overline{a}_t^s + q_t \overline{a}_t^b = a_t^m + p_t a_t^s \tag{139}$$

$$-\lambda \phi_t^s \overline{a}_t^s \leq \overline{a}_t^b. \tag{140}$$

Notice that the first-order condition with respect to k_t implies (31) so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{(\overline{a}_t^m, \overline{a}_t^s) \in \mathbb{R}_+^2, \overline{a}_t^b \in \mathbb{R}} \left\{ (\varepsilon y_t + \phi_t^s) \left[\overline{a}_t^s - \overline{a}_{10t}^s(\boldsymbol{a}_t, \varepsilon) \right] + \phi_t^m \left[\overline{a}_t^m - \overline{a}_{10t}^m(\boldsymbol{a}_t, \varepsilon) \right] + \overline{a}_t^b \right\}$$
s.t. (139), and (140).

Once the solution $\overline{a}_{11t}^m(\boldsymbol{a}_t, \varepsilon)$, $\overline{a}_{11t}^s(\boldsymbol{a}_t, \varepsilon)$, and $\overline{a}_{11t}^b(\boldsymbol{a}_t, \varepsilon)$ to this problem has been found, $k_{11t}(\boldsymbol{a}_t, \varepsilon)$ is given by (31). If we use (139) to substitute for $\overline{a}_t^b$, the auxiliary problem is equivalent to

$$\max_{(\overline{a}_t^m, \overline{a}_t^s) \in \mathbb{R}_+^2} \left[\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t \right) \overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t} \right) \overline{a}_t^m \right] \tag{141}$$

$$\text{s.t. } 0 \leq a_t^m + p_t a_t^s - \overline{a}_t^m - (p_t - \lambda q_t \phi_t^s) \overline{a}_t^s. \tag{142}$$

This problem has no solution if $p_t \leq \lambda q_t \phi_t^s$. To see this, assume $p_t \leq \lambda q_t \phi_t^s$. Set $\overline{a}_t^m = a_t^m + p_t a_t^s$ (a feasible choice), and notice (142) is satisfied by any $\overline{a}_t^s \in \mathbb{R}_+$. Thus the value of (141) is bounded below by

$$\left(\phi_t^m - \frac{1}{q_t} \right) (a_t^m + p_t a_t^s) + \max_{\overline{a}_t^s \in \mathbb{R}_+} \left[\varepsilon y_t + (1 - \lambda) \phi_t^s \right] \overline{a}_t^s,$$

which is arbitrarily large. Hence condition (27) is necessary for the bargaining problem to have as solution. The Lagrangian corresponding to the auxiliary problem (141) is

$$\begin{aligned} \mathcal{L} = & \left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t \right) \overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t} \right) \overline{a}_t^m \\ & + \xi^b \left[a_t^m + p_t a_t^s - \overline{a}_t^m - (p_t - \lambda q_t \phi_t^s) \overline{a}_t^s \right] + \xi^m \overline{a}_t^m + \xi^s \overline{a}_t^s, \end{aligned}$$

where ξ^b, ξ^m, and ξ^s are the multipliers on the constraints (142), $0 \leq \overline{a}_t^m$, and $0 \leq \overline{a}_t^s$, respectively. The first-order conditions are

$$\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t + \xi^s - (p_t - \lambda q_t \phi_t^s) \xi^b = 0$$

$$\phi_t^m - \frac{1}{q_t} + \xi^m - \xi^b = 0.$$

By working out the eight possible binding patterns for the multipliers $\left(\xi^b, \xi^m, \xi^s\right)$ and collecting the optimal allocations along with the inequality restrictions implied by each case, we obtain (28)-(31).

(ii). From (129), it is easy to show the solution to (2) is the same as the solution to (137).

(iii). The optimization problem (1) is the same as (3) with $\varepsilon = 0$. ■

A.2 Value functions

In this section we derive the value functions for brokers and investors, in a monetary economy (Lemma 3), and in a nonmonetary economy (Lemma 4).

Lemma 3 *Consider an economy with money. (i) The value function of a bond broker who enters the OTC round of period t with portfolio* $\boldsymbol{a}_t$ *is*

$$V_t^B\left(\boldsymbol{a}_t\right) = v_{Bt}^m a_t^m + v_{Bt}^s a_t^s + \Xi_t + \bar{W}_t^B, \tag{143}$$

where

$$\begin{aligned}
v_{Bt}^m &\equiv \frac{1}{q_t}\left[1 + \left(q_t\phi_t^m - 1\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\right] \\
v_{Bt}^s &\equiv \left[1 + \lambda\left(q_t\phi_t^m - 1\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\right]\phi_t^s \\
\Xi_t &\equiv \int\left[\alpha_{01}^B k_{01t}\left(\tilde{\boldsymbol{a}}_t, \varepsilon\right) + \alpha_{11}^B k_{11t}\left(\tilde{\boldsymbol{a}}_t, \varepsilon\right)\right] dH_{It}\left(\tilde{\boldsymbol{a}}_t, \varepsilon\right).
\end{aligned}$$

(ii) The value function of an equity broker who enters the OTC round of period t with portfolio $\boldsymbol{a}_t$ *is*

$$V_t^E\left(\boldsymbol{a}_t\right) = v_{Et}^m a_t^m + v_{Et}^s a_t^s + \bar{W}_t^E, \tag{144}$$

where

$$\begin{aligned}
v_{Et}^m &\equiv \phi_t^m - \frac{1}{p_t}\mathbb{I}_{\left\{\varepsilon_{10t}^*<0\right\}}\varepsilon_{10t}^* y_t \\
v_{Et}^s &\equiv p_t v_{Et}^m.
\end{aligned}$$

(iii) The value function of an investor who enters the OTC round of period t with portfolio $\boldsymbol{a}_t$ *and valuation* ε *is*

$$V_t^I\left(\boldsymbol{a}_t, \varepsilon\right) = v_{It}^m\left(\varepsilon\right) a_t^m + v_{It}^s\left(\varepsilon\right) a_t^s + \bar{W}_t^I, \tag{145}$$

where

$$\begin{aligned}
v_{It}^{m}(\varepsilon) &\equiv \phi_t^m + [\alpha_{10} + \alpha_{11}(1-\theta)]\,\mathbb{I}_{\{\varepsilon_{10t}^{*}<\varepsilon\}}(\varepsilon - \varepsilon_{10t}^{*})\,y_t\frac{1}{p_t} \\
&+ (\alpha_{01} + \alpha_{11})\,\theta\mathbb{I}_{\{q_t\phi_t^m<1\}}\left(\frac{1}{q_t} - \phi_t^m\right) \\
&+ \alpha_{11}\theta\mathbb{I}_{\{\varepsilon_{11t}^{*}<\varepsilon\}}(\varepsilon - \varepsilon_{11t}^{*})\,y_t\frac{1}{p_t - \lambda q_t\phi_t^s} \\
v_{It}^{s}(\varepsilon) &\equiv \varepsilon y_t + \phi_t^s + [\alpha_{10} + \alpha_{11}(1-\theta)]\,\mathbb{I}_{\{\varepsilon<\varepsilon_{10t}^{*}\}}(\varepsilon_{10t}^{*} - \varepsilon)\,y_t \\
&+ (\alpha_{01} + \alpha_{11})\,\theta\left(\phi_t^m - \frac{1}{q_t}\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\lambda q_t\phi_t^s \\
&+ \alpha_{11}\theta(\varepsilon - \varepsilon_{11t}^{*})\,y_t\frac{\lambda q_t\phi_t^s - \mathbb{I}_{\{\varepsilon<\varepsilon_{11t}^{*}\}}p_t}{p_t - \lambda q_t\phi_t^s}.
\end{aligned}$$

Proof. With (129), (130), and (128), the value function (10) becomes

$$V_t^B(\boldsymbol{a}_t) = \phi_t^m \bar{a}_{Bt}^m(\boldsymbol{a}_t) + \phi_t^s a_t^s + \bar{a}_{Bt}^b(\boldsymbol{a}_t) + \Xi_t + \bar{W}_t^B$$

with Ξ_t as defined in the statement, the value function (9) becomes

$$V_t^E(\boldsymbol{a}_t) = \phi_t^m \bar{a}_{Et}^m(\boldsymbol{a}_t) + \phi_t^s \bar{a}_{Et}^s(\boldsymbol{a}_t) + \bar{W}_t^E,$$

and the value function (11) becomes

$$\begin{aligned}
V_t^I(\boldsymbol{a}_t,\varepsilon) = \bar{W}_t^I &+ \alpha_{00}\left[(\varepsilon y_t + \phi_t^s)\,a_t^s + \phi_t^m a_t^m\right] \\
&+ \alpha_{10}\left[(\varepsilon y_t + \phi_t^s)\,\bar{a}_{10t}^s(\boldsymbol{a}_t,\varepsilon) + \phi_t^m \bar{a}_{10t}^m(\boldsymbol{a}_t,\varepsilon)\right] \\
&+ \alpha_{01}\left[(\varepsilon y_t + \phi_t^s)\,a_t^s + \phi_t^m \bar{a}_{01t}^m(\boldsymbol{a}_t,\varepsilon) + \bar{a}_{01t}^b(\boldsymbol{a}_t,\varepsilon) - k_{01t}(\boldsymbol{a}_t,\varepsilon)\right] \\
&+ \alpha_{11}\left[(\varepsilon y_t + \phi_t^s)\,\bar{a}_{11t}^s(\boldsymbol{a}_t,\varepsilon) + \phi_t^m \bar{a}_{11t}^m(\boldsymbol{a}_t,\varepsilon) + \bar{a}_{11t}^b(\boldsymbol{a}_t,\varepsilon) - k_{11t}(\boldsymbol{a}_t,\varepsilon)\right]. \qquad (146)
\end{aligned}$$

(*i*) With Lemma 2, $V_t^B(\boldsymbol{a}_t)$ can be written as (143). (*ii*) With Lemma 2, $V_t^E(\boldsymbol{a}_t)$ can be written as (144). (*iii*) Substitute $k_{11t}(\boldsymbol{a}_t,\varepsilon)$, $k_{01t}(\boldsymbol{a}_t,\varepsilon)$, $\bar{a}_{01t}^b(\boldsymbol{a}_t,\varepsilon)$, and $\bar{a}_{11t}^b(\boldsymbol{a}_t,\varepsilon)$ with (31), (138), $\bar{a}_{01t}^b(\boldsymbol{a}_t,\varepsilon) = -\frac{1}{q_t}\left[\bar{a}_{01t}^m(\boldsymbol{a}_t,\varepsilon) - a_t^m\right]$, and (30), respectively, to obtain

$$\begin{aligned}
V_t^I(\boldsymbol{a}_t,\varepsilon) = \bar{W}_t^I &+ (\varepsilon y_t + \phi_t^s)\,a_t^s + \phi_t^m a_t^m \\
&+ [\alpha_{10} + \alpha_{11}(1-\theta)]\left\{(\varepsilon y_t + \phi_t^s)\left[\bar{a}_{10t}^s(\boldsymbol{a}_t,\varepsilon) - a_t^s\right] + \phi_t^m\left[\bar{a}_{10t}^m(\boldsymbol{a}_t,\varepsilon) - a_t^m\right]\right\} \\
&+ \alpha_{01}\theta\left(\phi_t^m - \frac{1}{q_t}\right)\left[\bar{a}_{01t}^m(\boldsymbol{a}_t,\varepsilon) - a_t^m\right] \\
&+ \alpha_{11}\theta\left\{\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t\right)\left[\bar{a}_{11t}^s(\boldsymbol{a}_t,\varepsilon) - a_t^s\right] + \left(\phi_t^m - \frac{1}{q_t}\right)\left[\bar{a}_{11t}^m(\boldsymbol{a}_t,\varepsilon) - a_t^m\right]\right\}.
\end{aligned}$$

Then use Lemma 2 to replace the post-trade allocations $\bar{a}^{s}_{10t}(\boldsymbol{a}_t,\varepsilon)$, $\bar{a}^{m}_{10t}(\boldsymbol{a}_t,\varepsilon)$, $\bar{a}^{m}_{01t}(\boldsymbol{a}_t,\varepsilon)$, $\bar{a}^{s}_{11t}(\boldsymbol{a}_t,\varepsilon)$, and $\bar{a}^{m}_{11t}(\boldsymbol{a}_t,\varepsilon)$, and rearrange terms to arrive at (145). ■

Lemma 4 *Consider an economy without money. (i) The value function of a bond broker who enters the OTC round of period t with equity holding a^s_t is*

$$V^B_t(a^s_t) = \phi^s_t a^s_t + \Xi_t + \bar{W}^B_t, \tag{147}$$

where

$$\bar{W}^B_t \equiv \max_{\tilde{a}^s_{t+1}\in\mathbb{R}_+} \left[-\phi^s_t \tilde{a}^s_{t+1} + \beta\mathbb{E}_t V^B_{t+1}(\eta\tilde{a}^s_{t+1})\right] \tag{148}$$

and $\Xi_t \equiv \alpha^B_{11}\int k_{11t}(\tilde{a}^s_t,\varepsilon)\,dH_{It}(\tilde{a}^s_t,\varepsilon)$. (ii) The value function of an equity broker who enters the OTC round of period t with equity holding a^s_t is

$$V^E_t(a^s_t) = \phi^s_t a^s_t + \bar{W}^E_t, \tag{149}$$

where

$$\bar{W}^E_t \equiv \max_{\tilde{a}^s_{t+1}\in\mathbb{R}_+} \left[-\phi^s_t \tilde{a}^s_{t+1} + \beta\mathbb{E}_t V^E_{t+1}(\eta\tilde{a}^s_{t+1})\right]. \tag{150}$$

(iii) The value function of an investor who enters the OTC round of period t with equity holding a^s_t and valuation ε is

$$V^I_t(a^s_t,\varepsilon) = \left\{\varepsilon y_t + \phi^s_t + \alpha_{11}\theta(\varepsilon-\varepsilon^n_t)\,y_t\left[\chi(\varepsilon^n_t,\varepsilon)\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}-1\right]\right\}a^s_t + \bar{W}^I_t, \tag{151}$$

where

$$\bar{W}^I_t \equiv \max_{\tilde{a}^s_{t+1}\in\mathbb{R}_+} \left[-\phi^s_t \tilde{a}^s_{t+1} + \beta\mathbb{E}_t\int V^I_{t+1}\left[\eta\tilde{a}^s_{t+1}+(1-\eta)A^s,\varepsilon\right]dG(\varepsilon)\right]. \tag{152}$$

Proof. In a nonmonetary economy, (128)-(133) reduce to

$$W^I_t(a^s_t,a^b_t,k_t) = \phi^s_t a^s_t + a^b_t - k_t + \bar{W}^I_t \tag{153}$$
$$W^B_t(a^s_t,a^b_t,k_t) = \phi^s_t a^s_t + a^b_t + k_t + \bar{W}^B_t \tag{154}$$
$$W^E_t(a^s_t) = \phi^s_t a^s_t + \bar{W}^E_t, \tag{155}$$

where $\bar{W}^B_t$, $\bar{W}^E_t$, and $\bar{W}^I_t$ are given by (148), (150), and (152). (i) With (154) and Lemma 1, (10) reduces to (147). (ii) With (155) and Lemma 1, (9) reduces to (149). (iii) With (153), and Lemma 1, (11) reduces to (151). ■

A.3 Euler equations

In this section we derive the Euler equations that characterize the optimal portfolio choices in the second subperiod, in a monetary economy (Lemma 5) and in a nonmonetary economy (Lemma 6).

Lemma 5 *Consider an economy with money. Let* $\left(\tilde{a}^m_{kt+1}, \tilde{a}^s_{kt+1}\right)$ *denote the portfolio choice of an agent of type* $k \in \{B, E, I\}$ *in the second subperiod of period* t*. The portfolio* $\left(\tilde{a}^m_{kt+1}, \tilde{a}^s_{kt+1}\right)$ *is optimal for* $k \in \{B, E, I\}$ *if and only if it satisfies*

$$\left(\phi^m_t - \beta \mathbb{E}_t \bar{v}^m_{kt+1}\right) \tilde{a}^m_{kt+1} = 0 \leq \phi^m_t - \beta \mathbb{E}_t \bar{v}^m_{kt+1} \tag{156}$$

$$\left(\phi^s_t - \beta\eta \mathbb{E}_t \bar{v}^s_{kt+1}\right) \tilde{a}^s_{kt+1} = 0 \leq \phi^s_t - \beta\eta \mathbb{E}_t \bar{v}^s_{kt+1}, \tag{157}$$

where $\bar{v}^j_{kt+1} = v^j_{kt+1}$ *for* $k \in \{B, E\}$ *and* $j \in \{m, s\}$*,*

$$\begin{aligned}\bar{v}^m_{It+1} \equiv\ & \phi^m_{t+1} + (\alpha_{01} + \alpha_{11})\,\theta \left(\frac{1}{q_{t+1}} - \phi^m_{t+1}\right) \mathbb{I}_{\left\{q_{t+1}\phi^m_{t+1} < 1\right\}} \\ & + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right] \int_{\varepsilon^*_{10t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{10t+1}\right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon) \\ & + \alpha_{11}\theta \frac{1}{p_{t+1} - \lambda q_{t+1}\phi^s_{t+1}} \int_{\varepsilon^*_{11t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{11t+1}\right) y_{t+1} dG(\varepsilon),\end{aligned}$$

and

$$\begin{aligned}\bar{v}^s_{It+1} \equiv\ & \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (\alpha_{01} + \alpha_{11})\,\theta \left(\phi^m_{t+1} - \frac{1}{q_{t+1}}\right) \mathbb{I}_{\left\{1 < q_{t+1}\phi^m_{t+1}\right\}} \lambda q_{t+1}\phi^s_{t+1} \\ & + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right] \int_{\varepsilon_L}^{\varepsilon^*_{10t+1}} \left(\varepsilon^*_{10t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\ & + \alpha_{11}\theta \left[\int_{\varepsilon_L}^{\varepsilon^*_{11t+1}} \left(\varepsilon^*_{11t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1}\phi^s_{t+1}}{p_{t+1} - \lambda q_{t+1}\phi^s_{t+1}} \int_{\varepsilon^*_{11t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{11t+1}\right) y_{t+1} dG(\varepsilon)\right].\end{aligned}$$

Proof. With (143) and (132), the portfolio problem of a bond broker in the second subperiod can be written as

$$\bar{W}^B_t \equiv \beta \mathbb{E}_t \left(\Xi_{t+1} + \bar{W}^B_{t+1}\right) + \max_{(\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1}) \in \mathbb{R}^2_+} \left[\left(\beta \mathbb{E}_t v^m_{Bt+1} - \phi^m_t\right) \tilde{a}^m_{t+1} + \left(\beta\eta \mathbb{E}_t v^s_{Bt+1} - \phi^s_t\right) \tilde{a}^s_{t+1}\right].$$

With (144) and (133), the portfolio problem of an equity broker in the second subperiod can be written as

$$\bar{W}^E_t \equiv \beta \mathbb{E}_t \bar{W}^E_{t+1} + \max_{(\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1}) \in \mathbb{R}^2_+} \left[\left(\beta \mathbb{E}_t v^m_{Et+1} - \phi^m_t\right) \tilde{a}^m_{t+1} + \left(\beta\eta \mathbb{E}_t v^s_{Et+1} - \phi^s_t\right) \tilde{a}^s_{t+1}\right].$$

With (145) and (131), the portfolio problem of an investor in the second subperiod can be written as

$$\bar{W}_t^I \equiv T_t + \beta \mathbb{E}_t \left[\bar{W}_{t+1}^I + \bar{v}_{It+1}^s (1-\eta) A^s \right] + \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s) \in \mathbb{R}_+^2} \left[\left(\beta \mathbb{E}_t \bar{v}_{It+1}^m - \phi_t^m \right) \tilde{a}_{t+1}^m + \left(\beta \mathbb{E}_t \eta \bar{v}_{It+1}^s - \phi_t^s \right) \tilde{a}_{t+1}^s \right],$$

where $\bar{v}_{It+1}^k \equiv \int v_{It+1}^k (\varepsilon) \, dG(\varepsilon)$ for $k \in \{m, s\}$. ■

Lemma 6 *Consider an economy with no money. Let $\tilde{a}_{kt+1}^s$ denote equity holding chosen by an agent of type $k \in \{B, E, I\}$ in the second subperiod of period t. Then $\tilde{a}_{kt+1}^s$ is optimal if and only if it satisfies*

$$\left(\phi_t^s - \beta \eta \mathbb{E}_t \phi_{t+1}^s \right) \tilde{a}_{kt+1}^s = 0 \leq \phi_t^s - \beta \eta \mathbb{E}_t \phi_{t+1}^s, \text{ for } k \in \{B, E\} \tag{158}$$

and

$$-\phi_t^s + \beta \eta \mathbb{E}_t \left\{ \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon_{t+1}^n} \left(\varepsilon_{t+1}^n - \varepsilon \right) y_{t+1} dG(\varepsilon) + \frac{\lambda \phi_{t+1}^s}{\bar{\phi}_{t+1}^s - \lambda \phi_{t+1}^s} \int_{\varepsilon_{t+1}^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^n \right) y_{t+1} dG(\varepsilon) \right] \right\} \leq 0, \text{ with "=" if } \tilde{a}_{It+1}^s > 0. \tag{159}$$

Proof. With (147), (148), (149), and (150), the portfolio problem of a bond broker or of an equity broker in the second subperiod can be written as $\max_{\tilde{a}_{t+1}^s \in \mathbb{R}_+} \left[-\phi_t^s + \beta \eta \mathbb{E}_t \phi_{t+1}^s \right] \tilde{a}_{t+1}^s$. With (151) and (152), the portfolio problem of an investor in the second subperiod can be written as

$$\max_{\tilde{a}_{t+1}^s \in \mathbb{R}_+} \left[-\phi_t^s + \beta \eta \mathbb{E}_t \left\{ \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon_{t+1}^n} \left(\varepsilon_{t+1}^n - \varepsilon \right) y_{t+1} dG(\varepsilon) + \frac{\lambda \phi_{t+1}^s}{\bar{\phi}_{t+1}^s - \lambda \phi_{t+1}^s} \int_{\varepsilon_{t+1}^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^n \right) y_{t+1} dG(\varepsilon) \right] \right\} \right] \tilde{a}_{t+1}^s.$$

■

A.4 Market-clearing conditions

In this section we derive the market-clearing conditions for equity and bonds in the OTC round, in a monetary economy (Lemma 7) and in a nonmonetary economy (Lemma 8).

Lemma 7 *In a monetary equilibrium, the market-clearing conditions for equity,* $\bar{A}^s_{Et} + \bar{A}^s_{10t} + \bar{A}^s_{11t} = A^s_{Et} + (\alpha_{10} + \alpha_{11}) A^s_{It}$*, and bonds,* $\bar{A}^b_{Bt} + \bar{A}^b_{11t} + \bar{A}^b_{01t} = 0$*, in the OTC round are:*

$$\begin{aligned} 0 &= \alpha_{10}\left[1 - G\left(\varepsilon^*_{10t}\right)\right] \frac{A^m_{It} + p_t A^s_{It}}{p_t} + \alpha_{11}\left[1 - G\left(\varepsilon^*_{11t}\right)\right] \frac{A^m_{It} + p_t A^s_{It}}{p_t - \lambda q_t \phi^s_t} \\ &\quad - \left[A^s_{Et} + (\alpha_{10} + \alpha_{11}) A^s_{It}\right] \qquad (160) \\ 0 &= \left[1 - \chi\left(1, q_t\phi^m_t\right)\right] \frac{1}{q_t} A^m_{Bt} - \chi\left(1, q_t\phi^m_t\right) \lambda \phi^s_t A^s_{Bt} \\ &\quad + \alpha_{01}\left\{\left[1 - \chi\left(1, q_t\phi^m_t\right)\right] \frac{1}{q_t} A^m_{It} - \chi\left(1, q_t\phi^m_t\right) \lambda \phi^s_t A^s_{It}\right\} \\ &\quad + \alpha_{11}\Bigg\{ \left\{1 - \mathbb{I}_{\{1<q_t\phi^m_t\}} - \mathbb{I}_{\{q_t\phi^m_t=1\}}\left[1 - \chi\left(1, q_t\phi^m_t\right)\right]\right\} G\left(\varepsilon^*_{11t}\right) \\ &\quad - \frac{\lambda q_t \phi^s_t}{p_t - \lambda q_t \phi^s_t}\left[1 - G\left(\varepsilon^*_{11t}\right)\right] \Bigg\} \frac{A^m_{It} + p_t A^s_{It}}{q_t}. \qquad (161) \end{aligned}$$

Proof. By Lemma 2, the aggregate post-trade holdings of equity for agents who trade in the equity market in the OTC round of period t are

$$\begin{aligned} \bar{A}^s_{Et} &= N_E \int \bar{a}^s_{Et}\left(\boldsymbol{a}_t\right) dF_{Et}\left(\boldsymbol{a}_t\right) = \chi\left(\varepsilon^*_{10t}, 0\right) \frac{A^m_{Et} + p_t A^s_{Et}}{p_t} = 0 \\ \bar{A}^s_{11t} &= \alpha_{11} N_I \int \bar{a}^s_{11t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = \alpha_{11}\left[1 - G\left(\varepsilon^*_{11t}\right)\right] \frac{A^m_{It} + p_t A^s_{It}}{p_t - \lambda q_t \phi^s_t} \\ \bar{A}^s_{10t} &= \alpha_{10} N_I \int \bar{a}^s_{10t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = \alpha_{10}\left[1 - G\left(\varepsilon^*_{10t}\right)\right] \frac{A^m_{It} + p_t A^s_{It}}{p_t} \end{aligned}$$

and the the aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\begin{aligned} \bar{A}^b_{Bt} &= N_B \int \bar{a}^b_{Bt}\left(\boldsymbol{a}_t\right) dF_{Bt}\left(\boldsymbol{a}_t\right) = \left[1 - \chi\left(1, q_t\phi^m_t\right)\right] \frac{1}{q_t} A^m_{Bt} - \chi\left(1, q_t\phi^m_t\right) \lambda \phi^s_t A^s_{Bt} \\ \bar{A}^b_{11t} &= \alpha_{11} N_I \int \bar{a}^b_{11t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) \\ &= \alpha_{11}\Bigg\{ \left\{1 - \mathbb{I}_{\{1<q_t\phi^m_t\}} - \mathbb{I}_{\{q_t\phi^m_t=1\}}\left[1 - \chi\left(1, q_t\phi^m_t\right)\right]\right\} G\left(\varepsilon^*_{11t}\right) \\ &\quad - \frac{\lambda q_t \phi^s_t}{p_t - \lambda q_t \phi^s_t}\left[1 - G\left(\varepsilon^*_{11t}\right)\right] \Bigg\} \frac{A^m_{It} + p_t A^s_{It}}{q_t} \\ \bar{A}^b_{01t} &= \alpha_{01} N_I \int \bar{a}^b_{01t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = \alpha_{01}\left\{\left[1 - \chi\left(1, q_t\phi^m_t\right)\right] \frac{1}{q_t} A^m_{It} - \chi\left(1, q_t\phi^m_t\right) \lambda \phi^s_t A^s_{It}\right\}. \end{aligned}$$

■

Lemma 8 *In a nonmonetary equilibrium, the market-clearing condition for equity,* $\bar{A}^s_{Et}+\bar{A}^s_{10t}+\bar{A}^s_{11t}=A^s_{Et}+(\alpha_{10}+\alpha_{11})A^s_{It}$ *(or bonds,* $\bar{A}^b_{Bt}+\bar{A}^b_{11t}+\bar{A}^b_{01t}=0$*) in the OTC round is:*

$$1=[1-G(\varepsilon^n_t)]\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}. \tag{162}$$

Proof. By Lemma 1, the aggregate post-trade holdings of equity for agents who trade in the equity market in the OTC round of period t are

$$\begin{aligned}
\bar{A}^s_{Et}&=N_E\int\bar{a}^s_{Et}(a_t)\,dF_{Et}(a_t)=A^s_{Et}\\
\bar{A}^s_{11t}&=\alpha_{11}N_I\int\bar{a}^s_{11t}(a_t,\varepsilon)dH_{It}(a_t,\varepsilon)=\int\alpha_{11}\chi(\varepsilon^n_t,\varepsilon)\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}A^s_{It}dG(\varepsilon)\\
\bar{A}^s_{10t}&=\alpha_{10}N_I\int\bar{a}^s_{10t}(a_t,\varepsilon)dH_{It}(a_t,\varepsilon)=\alpha_{10}A^s_{It}
\end{aligned}$$

and the the aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\begin{aligned}
\bar{A}^b_{Bt}&=N_B\int\bar{a}^b_{Bt}(a_t)\,dF_{Bt}(a_t)=0\\
\bar{A}^b_{11t}&=\alpha_{11}N_I\int\bar{a}^b_{11t}(a_t,\varepsilon)dH_{It}(a_t,\varepsilon)=\int\alpha_{11}\bar{\phi}^s_t\left[1-\chi(\varepsilon^n_t,\varepsilon)\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}\right]A^s_{It}dG(\varepsilon)\\
\bar{A}^b_{01t}&=\alpha_{01}N_I\int\bar{a}^b_{01t}(a_t,\varepsilon)dH_{It}(a_t,\varepsilon)=0.
\end{aligned}$$

■

A.5 Equilibrium conditions

In this section we state the operational definitions of monetary and nonmonetary equilibrium that are used in the analysis.

A.5.1 Sequential nonmonetary equilibrium

Definition 4 *A (sequential) nonmonetary equilibrium is an allocation* $(\tilde{A}^s_{jt+1})_{j\in\{B,E,I\}}$ *and a sequence of prices,* $\{\phi^s_t,\bar{\phi}^s_t\}^{\infty}_{t=0}$*, that satisfy the three optimality conditions, (158) and (159) (with* $\tilde{a}^k_{jt+1}=\tilde{A}^k_{jt+1}$*), and the market-clearing conditions* $\tilde{A}^s_{Bt+1}+\tilde{A}^s_{Et+1}+\tilde{A}^s_{It+1}=A^s$ *and (162).*

Definition 4 follows from Definition 1 after recognizing that all agents of the same type $j\in\{B,E,I\}$ choose the same end-of-period portfolio that is characterized by the Euler equations derived in Lemma 6, and using the explicit version of the market clearing condition for equity and bonds in the OTC round derived in Lemma 8. Given the equilibrium objects in Definition 4, the bargaining outcomes, which are part of Definition 1 but not Definition 4, are immediate from Lemma 1.

According to Definition 4, a nonmonetary equilibrium can be characterized by sequence of prices, $\{\phi_t^s, \bar{\phi}_t^s\}_{t=0}^{\infty}$ and an allocation $(\tilde{A}_{jt+1}^s)_{j\in\{B,E,I\}}$ that satisfy the market-clearing conditions

$$A^s = \tilde{A}_{Bt+1}^s + \tilde{A}_{Et+1}^s + \tilde{A}_{It+1}^s \tag{163}$$

$$1 = [1 - G(\varepsilon_t^n)] \frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \lambda \phi_t^s} \tag{164}$$

and the optimality conditions

$$\left(\phi_t^s - \beta\eta\mathbb{E}_t\phi_{t+1}^s\right)\tilde{A}_{jt+1}^s = 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\phi_{t+1}^s, \text{ for } j \in \{B, E\} \tag{165}$$

and

$$\begin{aligned} &-\phi_t^s + \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + \alpha_{11}\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon_{t+1}^n}\left(\varepsilon_{t+1}^n - \varepsilon\right)y_{t+1}dG(\varepsilon) \\ &+ \frac{\lambda\phi_{t+1}^s}{\bar{\phi}_{t+1}^s - \lambda\phi_{t+1}^s}\int_{\varepsilon_{t+1}^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^n\right)y_{t+1}dG(\varepsilon)\Bigg]\Bigg\} \\ &\leq 0, \text{ with} \text{“} = \text{”} \text{ if } \tilde{A}_{It+1}^s > 0, \end{aligned} \tag{166}$$

where ε_t^n is given by (13).

A.5.2 Recursive nonmonetary equilibrium

The following result summarizes the conditions that characterize a recursive nonmonetary equilibrium (RNE).

Lemma 9 *A recursive nonmonetary equilibrium is a vector* $(\varepsilon^n, \phi^s, (\tilde{A}_k^s)_{k\in\{B,E,I\}})$ *that satisfies the following conditions*

$$0 = \tilde{A}_B^s + \tilde{A}_E^s + \tilde{A}_I^s - A^s$$
$$1 = [1 - G(\varepsilon^n)] \frac{\varepsilon^n + \phi^s}{\varepsilon^n + (1-\lambda)\phi^s}$$
$$\phi^s \geq \bar{\beta}\eta\left\{\bar{\varepsilon} + \phi^s + \alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda\phi^s}{\varepsilon^n + (1-\lambda)\phi^s}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right]\right\}$$

with “=” if $\tilde{A}_I^s > 0$*, and* $\left(1 - \bar{\beta}\eta\right)\tilde{A}_k^s = 0$ *for* $k \in \{B, E\}$.

Proof. The equilibrium conditions in the statement of the lemma are obtained from (163)-(166) by using $\phi_t^s = \phi^s y_t$, $\bar{\phi}_t^s = \bar{\phi}^s y_t$, $A_{jt}^s = A_j^s$ for $j \in \{B, E, I\}$, and $\varepsilon_t^n = \left(\bar{\phi}_t^s - \phi_t^s\right) \frac{1}{y_t} = \bar{\phi}^s - \phi^s \equiv \varepsilon^n$. ■

The first equation in the statement of Lemma 9 is the second-subperiod market-clearing condition for equity. The second equation is the first-subperiod market-clearing condition for equity (or bonds). The remaining three conditions are the Euler equations for equity corresponding to investors, bond brokers, and equity brokers, respectively.

A.5.3 Sequential monetary equilibrium

Definition 5 *A (sequential) monetary equilibrium is an allocation* $\left((\tilde{A}_{jt+1}^k)_{k \in \{m,s\}}\right)_{j \in \{B,E,I\}}$ *and a sequence of prices,* $\{p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$*, that satisfy the six optimality conditions, (156)-(157) (with* $\tilde{a}_{jt+1}^k = \tilde{A}_{jt+1}^k$*), and the four market-clearing conditions,* $\tilde{A}_{Bt+1}^s + \tilde{A}_{Et+1}^s + \tilde{A}_{It+1}^s = A^s$*,* $\tilde{A}_{Bt+1}^m + \tilde{A}_{Et+1}^m + \tilde{A}_{It+1}^m = A_{t+1}^m$*, (160) and (161).*

Definition 5 follows from Definition 1 after recognizing that all agents of the same type $j \in \{B, E, I\}$ choose the same end-of-period portfolio that is characterized by the Euler equations derived in Lemma 5, and using the explicit version of the market clearing condition for equity in the OTC round derived in Lemma 7. Given the equilibrium objects in Definition 5, the bargaining outcomes, which are part of Definition 1 but not Definition 5, are immediate from Lemma 2.

According to Definition 5, a monetary equilibrium can be characterized by sequence of prices, $\{p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$ and an allocation $\left((\tilde{A}_{it+1}^k, \tilde{A}_{jt+1}^k)_{k \in \{m,s\}}\right)_{j \in \{B,E,I\}}$ that satisfy the following market-clearing conditions

$$
\begin{aligned}
0 &= \tilde{A}_{Bt+1}^s + \tilde{A}_{Et+1}^s + \tilde{A}_{It+1}^s - A^s \\
0 &= \tilde{A}_{Bt+1}^m + \tilde{A}_{Et+1}^m + \tilde{A}_{It+1}^m - A_{t+1}^m \\
0 &= \alpha_{10} \left[1 - G\left(\varepsilon_{10t}^*\right)\right] \frac{A_{It}^m + p_t A_{It}^s}{p_t} + \alpha_{11} \left[1 - G\left(\varepsilon_{11t}^*\right)\right] \frac{A_{It}^m + p_t A_{It}^s}{p_t - \lambda q_t \phi_t^s} - \left[A_{Et}^s + \left(\alpha_{10} + \alpha_{11}\right) A_{It}^s\right] \\
0 &= \left[1 - \chi\left(1, q_t \phi_t^m\right)\right] \frac{1}{q_t} A_{Bt}^m - \chi\left(1, q_t \phi_t^m\right) \lambda \phi_t^s A_{Bt}^s \\
&\quad + \alpha_{01} \left\{\left[1 - \chi\left(1, q_t \phi_t^m\right)\right] \frac{1}{q_t} A_{It}^m - \chi\left(1, q_t \phi_t^m\right) \lambda \phi_t^s A_{It}^s\right\} \\
&\quad + \alpha_{11} \left\{\left[1 - \mathbb{I}_{\{1 < q_t \phi_t^m\}} - \mathbb{I}_{\{q_t \phi_t^m = 1\}} \left(1 - \chi_{11}\right)\right] G\left(\varepsilon_{11t}^*\right) - \frac{\lambda q_t \phi_t^s}{p_t - \lambda q_t \phi_t^s} \left[1 - G\left(\varepsilon_{11t}^*\right)\right]\right\} \frac{A_{It}^m + p_t A_{It}^s}{q_t}
\end{aligned}
$$

and optimality conditions

$$\left(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{Bt+1}^m\right) \tilde{A}_{Bt+1}^m = 0 \leq \phi_t^m - \beta \mathbb{E}_t \bar{v}_{Bt+1}^m$$
$$\left(\phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{Bt+1}^s\right) \tilde{A}_{Bt+1}^s = 0 \leq \phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{Bt+1}^s$$
$$\left(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{Et+1}^m\right) \tilde{A}_{Et+1}^m = 0 \leq \phi_t^m - \beta \mathbb{E}_t \bar{v}_{Et+1}^m$$
$$\left(\phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{Et+1}^s\right) \tilde{A}_{Et+1}^s = 0 \leq \phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{Et+1}^s$$
$$\left(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m\right) \tilde{A}_{It+1}^m = 0 \leq \phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m$$
$$\left(\phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{It+1}^s\right) \tilde{A}_{It+1}^s = 0 \leq \phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{It+1}^s$$

where ε_{10t}^* is given by (19), ε_{11t}^* is given by (20), $i_t^m \equiv \frac{1}{q_t \phi_t^m} - 1$,

$$\bar{v}_{Bt+1}^m \equiv \frac{1}{q_{t+1}} \left(1 - \frac{i_{t+1}^m}{1 + i_{t+1}^m} \mathbb{I}_{\left\{i_{t+1}^m < 0\right\}}\right)$$
$$\bar{v}_{Bt+1}^s \equiv \left(1 - \lambda \frac{i_{t+1}^m}{1 + i_{t+1}^m} \mathbb{I}_{\left\{i_{t+1}^m < 0\right\}}\right) \phi_{t+1}^s$$
$$\bar{v}_{Et+1}^m \equiv \phi_{t+1}^m - \frac{1}{p_{t+1}} \varepsilon_{10t+1}^* y_{t+1} \mathbb{I}_{\left\{\varepsilon_{10t+1}^* < 0\right\}}$$
$$\bar{v}_{Et+1}^s \equiv p_{t+1} \bar{v}_{Et+1}^m,$$

and

$$\begin{aligned}
\bar{v}_{It+1}^m \equiv\ & \phi_{t+1}^m + \left(\alpha_{01} + \alpha_{11}\right) \theta \left(\frac{1}{q_{t+1}} - \phi_{t+1}^m\right) \mathbb{I}_{\left\{q_{t+1}\phi_{t+1}^m < 1\right\}} \\
& + \left[\alpha_{10} + \alpha_{11}\left(1 - \theta\right)\right] \int_{\varepsilon_{10t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{10t+1}^*\right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon) \\
& + \alpha_{11}\theta \frac{1}{p_{t+1} - \lambda q_{t+1} \phi_{t+1}^s} \int_{\varepsilon_{11t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11t+1}^*\right) y_{t+1} dG(\varepsilon) \\
\bar{v}_{It+1}^s \equiv\ & \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + \left(\alpha_{01} + \alpha_{11}\right) \theta \left(\phi_{t+1}^m - \frac{1}{q_{t+1}}\right) \mathbb{I}_{\left\{1 < q_{t+1}\phi_{t+1}^m\right\}} \lambda q_t \phi_{t+1}^s \\
& + \left[\alpha_{10} + \alpha_{11}\left(1 - \theta\right)\right] \int_{\varepsilon_L}^{\varepsilon_{10t+1}^*} \left(\varepsilon_{10t+1}^* - \varepsilon\right) y_{t+1} dG\left(\varepsilon\right) \\
& + \alpha_{11}\theta \left[\int_{\varepsilon_L}^{\varepsilon_{11t+1}^*} \left(\varepsilon_{11t+1}^* - \varepsilon\right) y_{t+1} dG\left(\varepsilon\right) + \frac{\lambda q_{t+1} \phi_{t+1}^s}{p_{t+1} - \lambda q_{t+1} \phi_{t+1}^s} \int_{\varepsilon_{11t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11t+1}^*\right) y_{t+1} dG\left(\varepsilon\right)\right].
\end{aligned}$$

A.5.4 Sequential monetary equilibrium with credit

The following result states that the credit market would be inactive if the net nominal interest rate on bonds, $i_t^m \equiv \frac{1}{q_t \phi_t^m} - 1$, were negative.

Lemma 10 *Consider a monetary equilibrium. If the bond market is active in period t, then $q_t\phi_t^m \leq 1$.*

Proof. In an equilibrium with $1 < q_t\phi_t^m$, the market-clearing condition (161) becomes

$$0 = \lambda\phi_t^s \left(A_{Bt}^s + \alpha_{01}A_{It}^s\right) + \alpha_{11}\frac{\lambda\phi_t^s}{p_t - \lambda q_t\phi_t^s}\left[1 - G\left(\varepsilon_{11t}^*\right)\right]\left(A_{It}^m + p_t A_{It}^s\right).$$

This condition can only hold if $A_{Bt}^s = A_{It}^s = \left[1 - G\left(\varepsilon_{11t}^*\right)\right]\left(A_{It}^m + p_t A_{It}^s\right) = 0$, i.e., if the bond market is inactive. The condition $1 < q_t\phi_t^m$ implies bond demand is nil, so the bond market can only clear with no trade. ■

According to Lemma 10, a monetary equilibrium with an active bond market can be characterized by sequence of prices, $\{p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$ and an allocation $\left((\tilde{A}_{it+1}^k, \tilde{A}_{jt+1}^k)_{k\in\{m,s\}}\right)_{j\in\{B,E,I\}}$ that satisfy the following market-clearing conditions

$$\begin{aligned}
0 &= \tilde{A}_{Bt+1}^s + \tilde{A}_{Et+1}^s + \tilde{A}_{It+1}^s - A^s \\
0 &= \tilde{A}_{Bt+1}^m + \tilde{A}_{Et+1}^m + \tilde{A}_{It+1}^m - A_{t+1}^m \\
0 &= \alpha_{10}\left[1 - G\left(\varepsilon_{10t}^*\right)\right]\frac{A_{It}^m + p_t A_{It}^s}{p_t} + \alpha_{11}\left[1 - G\left(\varepsilon_{11t}^*\right)\right]\frac{A_{It}^m + p_t A_{It}^s}{p_t - \lambda q_t\phi_t^s} - \left[A_{Et}^s + \left(\alpha_{10} + \alpha_{11}\right)A_{It}^s\right] \\
0 &= \left(1 - \mathbb{I}_{\{q_t\phi_t^m = 1\}}\chi_B\right)\frac{1}{q_t}A_{Bt}^m \\
&\quad + \alpha_{01}\left\{\left(1 - \chi_{01}\mathbb{I}_{\{q_t\phi_t^m = 1\}}\right)\frac{1}{q_t}A_{It}^m - \mathbb{I}_{\{q_t\phi_t^m = 1\}}\chi_{01}\lambda\phi_t^s A_{It}^s\right\} \\
&\quad + \alpha_{11}\left\{\left[1 - \mathbb{I}_{\{q_t\phi_t^m = 1\}}\left(1 - \chi_{11}\right)\right]G\left(\varepsilon_{11t}^*\right) - \frac{\lambda q_t\phi_t^s}{p_t - \lambda q_t\phi_t^s}\left[1 - G\left(\varepsilon_{11t}^*\right)\right]\right\}\frac{A_{It}^m + p_t A_{It}^s}{q_t}
\end{aligned}$$

and optimality conditions

$$\begin{aligned}
\left(\phi_t^m - \beta\mathbb{E}_t\frac{1}{q_{t+1}}\right)\tilde{A}_{Bt+1}^m &= 0 \leq \phi_t^m - \beta\mathbb{E}_t\frac{1}{q_{t+1}} \\
\left(\phi_t^s - \beta\eta\mathbb{E}_t\phi_{t+1}^s\right)\tilde{A}_{Bt+1}^s &= 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\phi_{t+1}^s \\
\left(\phi_t^m - \beta\mathbb{E}_t\bar{v}_{Et+1}^m\right)\tilde{A}_{Et+1}^m &= 0 \leq \phi_t^m - \beta\mathbb{E}_t\bar{v}_{Et+1}^m \\
\left(\phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{Et+1}^s\right)\tilde{A}_{Et+1}^s &= 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{Et+1}^s \\
\left(\phi_t^m - \beta\mathbb{E}_t\bar{v}_{It+1}^m\right)\tilde{A}_{It+1}^m &= 0 \leq \phi_t^m - \beta\mathbb{E}_t\bar{v}_{It+1}^m \\
\left(\phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{It+1}^s\right)\tilde{A}_{It+1}^s &= 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{It+1}^s
\end{aligned}$$

where

$$\bar{v}^m_{Et+1} \equiv \phi^m_{t+1} - \frac{1}{p_{t+1}} \varepsilon^*_{10t+1} y_{t+1} \mathbb{I}_{\{\varepsilon^*_{10t+1}<0\}}$$
$$\bar{v}^s_{Et+1} \equiv p_{t+1} \bar{v}^m_{Et+1}$$

and

$$\begin{aligned}
\bar{v}^m_{It+1} \equiv\ & \phi^m_{t+1} + (\alpha_{01} + \alpha_{11})\, \theta \left(\frac{1}{q_{t+1}} - \phi^m_{t+1} \right) \\
& + [\alpha_{10} + \alpha_{11} (1-\theta)] \int_{\varepsilon^*_{10t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{10t+1}\right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon) \\
& + \alpha_{11} \theta \frac{1}{p_{t+1} - \lambda q_{t+1} \phi^s_{t+1}} \int_{\varepsilon^*_{11t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{11t+1}\right) y_{t+1} dG(\varepsilon) \\
\bar{v}^s_{It+1} \equiv\ & \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} \\
& + [\alpha_{10} + \alpha_{11} (1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*_{10t+1}} \left(\varepsilon^*_{10t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
& + \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon^*_{11t+1}} \left(\varepsilon^*_{11t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1} \phi^s_{t+1}}{p_{t+1} - \lambda q_{t+1} \phi^s_{t+1}} \int_{\varepsilon^*_{11t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^*_{11t+1}\right) y_{t+1} dG(\varepsilon) \right].
\end{aligned}$$

A.5.5 Recursive monetary equilibrium with credit

The following result summarizes the conditions that characterize a recursive monetary equilibrium (RME).

Lemma 11 *A recursive monetary equilibrium is a vector* $(\varepsilon^*_{10}, \varepsilon^*_{11}, \phi^s, Z, (\tilde{A}^s_k, Z_k)_{k\in\{B,E,I\}})$ *that satisfies the following conditions*

$$\begin{aligned}
0 =\ & \tilde{A}^s_B + \tilde{A}^s_E + \tilde{A}^s_I - A^s \\
0 =\ & Z_B + Z_E + Z_I - Z \\
0 =\ & \left\{ \alpha_{10} \left[1 - G(\varepsilon^*_{10})\right] + \alpha_{11} \left[1 - G(\varepsilon^*_{11})\right] \frac{\varepsilon^*_{11} + \phi^s}{\varepsilon^*_{11} + (1-\lambda)\phi^s} \right\} \left(\frac{Z_I A^s}{\varepsilon^*_{10} + \phi^s} + A^s_I \right) \\
& - \left[A^s_E + (\alpha_{10} + \alpha_{11}) A^s_I\right] \\
0 =\ & \left(1 - \mathbb{I}_{\{\varepsilon^*_{10} = \varepsilon^*_{11}\}} \chi_B\right) Z_B A^s + \alpha_{01} \left\{ \left(1 - \chi_{01} \mathbb{I}_{\{\varepsilon^*_{10} = \varepsilon^*_{11}\}}\right) Z_I A^s - \mathbb{I}_{\{\varepsilon^*_{10} = \varepsilon^*_{11}\}} \chi_{01} \frac{\varepsilon^*_{10} + \phi^s}{\varepsilon^*_{11} + \phi^s} \lambda \phi^s A^s_I \right\} \\
& + \alpha_{11} \left\{ G(\varepsilon^*_{11}) \left[1 - \mathbb{I}_{\{\varepsilon^*_{10} = \varepsilon^*_{11}\}} (1 - \chi_{11})\right] - \left[1 - G(\varepsilon^*_{11})\right] \frac{\lambda \phi^s}{\varepsilon^*_{11} + (1-\lambda)\phi^s} \right\} \left[Z_I A^s + (\varepsilon^*_{10} + \phi^s) A^s_I\right]
\end{aligned}$$

$$\left(1-\frac{\bar{\beta}}{\mu}\frac{\varepsilon_{11}^{*}+\phi^{s}}{\varepsilon_{10}^{*}+\phi^{s}}\right)Z_{B}=0\leq\left(1-\frac{\bar{\beta}}{\mu}\frac{\varepsilon_{11}^{*}+\phi^{s}}{\varepsilon_{10}^{*}+\phi^{s}}\right)Z$$

$$\left(1-\bar{\beta}\eta\right)\tilde{A}_{B}^{s}=0\leq1-\bar{\beta}\eta$$

$$\left\{1-\frac{\bar{\beta}}{\mu}\left[1-\frac{\varepsilon_{10}^{*}\mathbb{I}_{\{\varepsilon_{10}^{*}<0\}}}{\varepsilon_{10}^{*}+\phi^{s}}\right]\right\}Z_{E}=0\leq\left\{1-\frac{\bar{\beta}}{\mu}\left[1-\frac{\varepsilon_{10}^{*}\mathbb{I}_{\{\varepsilon_{10}^{*}<0\}}}{\varepsilon_{10}^{*}+\phi^{s}}\right]\right\}Z$$

$$\left[\phi^{s}-\bar{\beta}\eta\left(\mathbb{I}_{\{0<\varepsilon_{10}^{*}\}}\varepsilon_{10}^{*}+\phi^{s}\right)\right]\tilde{A}_{E}^{s}=0\leq\phi^{s}-\bar{\beta}\eta\left(\mathbb{I}_{\{0<\varepsilon_{10}^{*}\}}\varepsilon_{10}^{*}+\phi^{s}\right)$$

$$\Gamma_{I}^{m}Z_{I}=0\leq\Gamma_{I}^{m}Z$$

$$\Gamma_{I}^{s}\tilde{A}_{I}^{s}=0\leq\Gamma_{I}^{s},$$

where $\chi_{B},\chi_{01},\chi_{11}\in[0,1]$, *and*

$$\Gamma_{I}^{m}\equiv\left\{1-\frac{\bar{\beta}}{\mu}\left[1+\left(\alpha_{01}+\alpha_{11}\right)\theta\left(\frac{\varepsilon_{11}^{*}+\phi^{s}}{\varepsilon_{10}^{*}+\phi^{s}}-1\right)\right.\right.$$
$$+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\frac{1}{\varepsilon_{10}^{*}+\phi^{s}}\int_{\varepsilon_{10}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{10}^{*}\right)dG(\varepsilon)$$
$$\left.\left.+\alpha_{11}\theta\frac{\varepsilon_{11}^{*}+\phi^{s}}{\varepsilon_{10}^{*}+\phi^{s}}\frac{1}{\varepsilon_{11}^{*}+\left(1-\lambda\right)\phi^{s}}\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG(\varepsilon)\right]\right\}$$

$$\Gamma_{I}^{s}\equiv\left\{\phi^{s}-\bar{\beta}\eta\left\{\bar{\varepsilon}+\phi^{s}+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{L}}^{\varepsilon_{10}^{*}}\left(\varepsilon_{10}^{*}-\varepsilon\right)dG\left(\varepsilon\right)\right.\right.$$
$$\left.\left.+\alpha_{11}\theta\left[\int_{\varepsilon_{L}}^{\varepsilon_{11}^{*}}\left(\varepsilon_{11}^{*}-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda\phi^{s}}{\varepsilon_{11}^{*}+\left(1-\lambda\right)\phi^{s}}\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG\left(\varepsilon\right)\right]\right\}\right\}.$$

with $A_{j}^{s}=\eta\tilde{A}_{j}^{s}$ *for* $j\in\{B,E\}$, $A_{I}^{s}=\eta\tilde{A}_{I}^{s}+\left(1-\eta\right)A^{s}$, *and* $Z>0$.

Proof. The equilibrium conditions in the statement of the lemma are obtained from the ones in Section A.5.4 by using $\phi_{t}^{s}=\phi^{s}y_{t}$, $p_{t}\phi_{t}^{m}\equiv\bar{\phi}_{10t}^{s}=\bar{\phi}_{10}^{s}y_{t}$, $p_{t}/q_{t}\equiv\bar{\phi}_{11t}^{s}=\bar{\phi}_{11}^{s}y_{t}$, $A_{jt}^{s}=A_{j}^{s}$ for $j\in\{B,E,I\}$, $\phi_{t}^{m}A_{t}^{m}=ZA^{s}y_{t}$, $\phi_{t}^{m}A_{jt}^{m}=Z_{j}A^{s}y_{t}$, for $j\in\{B,E,I\}$, $\varepsilon_{10t}^{*}=\left(p_{t}\phi_{t}^{m}-\phi_{t}^{s}\right)\frac{1}{y_{t}}=\bar{\phi}_{10}^{s}-\phi^{s}\equiv\varepsilon_{10}^{*}$, $\varepsilon_{11t}^{*}=\left(p_{t}/q_{t}-\phi_{t}^{s}\right)\frac{1}{y_{t}}=\bar{\phi}_{11}^{s}-\phi^{s}\equiv\varepsilon_{11}^{*}$, $p_{t}=\frac{\left(\varepsilon_{10}^{*}+\phi^{s}\right)A_{t}^{m}}{ZA^{s}}$, $\phi_{t}^{m}=\frac{ZA^{s}y_{t}}{A_{t}^{m}}$, $q_{t}=\frac{\left(\varepsilon_{10}^{*}+\phi^{s}\right)A_{t}^{m}}{\left(\varepsilon_{11}^{*}+\phi^{s}\right)ZA^{s}y_{t}}$, $\phi_{t+1}^{s}/\phi_{t}^{s}=\bar{\phi}_{10t+1}^{s}/\bar{\phi}_{10t}^{s}=\bar{\phi}_{11t+1}^{s}/\bar{\phi}_{11t}^{s}=\gamma_{t+1}$, $p_{t+1}/p_{t}=\mu$, and $\phi_{t}^{m}/\phi_{t+1}^{m}=q_{t+1}/q_{t}=\mu/\gamma_{t+1}$. ■

The first and second equations in Lemma 11 are the second-subperiod market-clearing conditions for equity and money, respectively. The third and fourth equations are the first-subperiod market-clearing condition for equity and bonds, respectively. The remaining six conditions are the Euler equations for money and equity, corresponding to bond brokers, equity brokers, and investors, respectively.

Corollary 6 *Consider the economy where brokers do not hold assets overnight. A recursive monetary equilibrium (with credit) is a vector* $(\varepsilon_{10}^*, \varepsilon_{11}^*, \phi^s, Z)$ *that satisfies*

$$
\begin{aligned}
0 &= \left\{\alpha_{10}\left[1-G\left(\varepsilon_{10}^*\right)\right]+\alpha_{11}\left[1-G\left(\varepsilon_{11}^*\right)\right]\frac{\varepsilon_{11}^*+\phi^s}{\varepsilon_{11}^*+(1-\lambda)\,\phi^s}\right\}\left(\frac{Z}{\varepsilon_{10}^*+\phi^s}+1\right)-(\alpha_{10}+\alpha_{11})\\
0 &= \alpha_{01}\left[\left(1-\chi_{01}\mathbb{I}_{\{\varepsilon_{10}^*=\varepsilon_{11}^*\}}\right)Z-\mathbb{I}_{\{\varepsilon_{10}^*=\varepsilon_{11}^*\}}\chi_{01}\frac{\varepsilon_{10}^*+\phi^s}{\varepsilon_{11}^*+\phi^s}\lambda\phi^s\right]\\
&\quad+\alpha_{11}\left\{G\left(\varepsilon_{11}^*\right)\left[1-\mathbb{I}_{\{\varepsilon_{10}^*=\varepsilon_{11}^*\}}\left(1-\chi_{11}\right)\right]-\left[1-G\left(\varepsilon_{11}^*\right)\right]\frac{\lambda\phi^s}{\varepsilon_{11}^*+(1-\lambda)\,\phi^s}\right\}\left(Z+\varepsilon_{10}^*+\phi^s\right)
\end{aligned}
$$

where $\chi_{01}, \chi_{11} \in [0,1]$*, and*

$$
\begin{aligned}
i^p &= (\alpha_{01}+\alpha_{11})\,\theta\left(\frac{\varepsilon_{11}^*+\phi^s}{\varepsilon_{10}^*+\phi^s}-1\right)+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\frac{1}{\varepsilon_{10}^*+\phi^s}\int_{\varepsilon_{10}^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{10}^*\right)dG(\varepsilon)\\
&\quad+\alpha_{11}\theta\frac{\varepsilon_{11}^*+\phi^s}{\varepsilon_{10}^*+\phi^s}\frac{1}{\varepsilon_{11}^*+(1-\lambda)\,\phi^s}\int_{\varepsilon_{11}^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{11}^*\right)dG(\varepsilon)\\
\frac{1-\bar{\beta}\eta}{\bar{\beta}\eta}\phi^s &= \bar{\varepsilon}+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_L}^{\varepsilon_{10}^*}\left(\varepsilon_{10}^*-\varepsilon\right)dG\left(\varepsilon\right)\\
&\quad+\alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon_{11}^*}\left(\varepsilon_{11}^*-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda\phi^s}{\varepsilon_{11}^*+(1-\lambda)\,\phi^s}\int_{\varepsilon_{11}^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{11}^*\right)dG\left(\varepsilon\right)\right].
\end{aligned}
$$

A.6 Continuous-time limiting economy

In this section we derive the equilibrium conditions for the continuous-time limiting economy (as $\Delta \to 0$) in which brokers are assumed not hold assets overnight.

A.6.1 Equilibrium conditions

Lemma 12 *Consider the limiting economy (as* $\Delta \to 0$*) where brokers do not hold assets overnight. A recursive nonmonetary equilibrium is a pair* (ε^n, φ) *that satisfies*

$$
\begin{aligned}
1 &= \frac{1-G\left(\varepsilon^n\right)}{1-\lambda}\\
\varphi &= \bar{\varepsilon}+\alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG(\varepsilon)\right].
\end{aligned}
$$

Proof. From Lemma 9, if the period length is Δ, an equilibrium for the economy where brokers do not hold assets overnight is a pair $(\varepsilon^n, \Phi^s(\Delta))$ that satisfies

$$1 = [1 - G(\varepsilon^n)] \frac{\varepsilon^n + \Phi^s(\Delta)}{\varepsilon^n + (1-\lambda)\Phi^s(\Delta)}$$

$$\Phi^s(\Delta) = \bar{\beta}\eta \Bigg\{ \bar{\varepsilon} + \Phi^s(\Delta) + \alpha_{11}\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \Phi^s(\Delta)}{\varepsilon^n + (1-\lambda)\Phi^s(\Delta)} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \Bigg] \Bigg\}$$

This can be written as

$$1 = [1 - G(\varepsilon^n)] \frac{\varepsilon^n \Delta + \Phi^s(\Delta)\Delta}{\varepsilon^n \Delta + (1-\lambda)\Phi^s(\Delta)\Delta}$$

$$\frac{r + \delta - g + g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)} \Phi^s(\Delta)\Delta = \bar{\varepsilon} + \alpha_{11}\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \Phi^s(\Delta)\Delta}{\varepsilon^n \Delta + (1-\lambda)\Phi^s(\Delta)\Delta} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \Bigg]$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ■

Lemma 13 *Consider the limiting economy (as $\Delta \to 0$) where brokers do not hold assets overnight. A recursive monetary equilibrium (with credit) is a vector $(\varepsilon_{10}^*, \varepsilon_{11}^*, \varphi, \mathcal{Z})$ that satisfies*

$$0 = \left\{ \alpha_{10}[1 - G(\varepsilon_{10}^*)] + \alpha_{11}[1 - G(\varepsilon_{11}^*)] \frac{1}{1-\lambda} \right\} \left(\frac{\mathcal{Z}}{\varphi} + 1 \right) - (\alpha_{10} + \alpha_{11})$$

$$0 = \alpha_{01} \left[\left(1 - \chi_{01} \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \right) \frac{\mathcal{Z}}{\varphi} - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \chi_{01} \lambda \right] + \alpha_{11} \left\{ G(\varepsilon_{11}^*) \left[1 - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} (1 - \chi_{11}) \right] - [1 - G(\varepsilon_{11}^*)] \frac{\lambda}{1-\lambda} \right\} \left(\frac{\mathcal{Z}}{\varphi} + 1 \right)$$

where $\chi_{01}, \chi_{11} \in [0,1]$, and

$$\iota\varphi = (\alpha_{01} + \alpha_{11})\theta(\varepsilon_{11}^* - \varepsilon_{10}^*) + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) + \alpha_{11}\theta \frac{1}{1-\lambda} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon)$$

$$\varphi = \bar{\varepsilon} + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta \left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \right].$$

Proof. If the period length is Δ, the equilibrium conditions in Corollary 6 generalize to

$$0 = \left\{ \alpha_{10} \left[1 - G\left(\varepsilon_{10}^*\right)\right] + \alpha_{11} \left[1 - G\left(\varepsilon_{11}^*\right)\right] \frac{\varepsilon_{11}^* + \Phi^s\left(\Delta\right)}{\varepsilon_{11}^* + \left(1-\lambda\right)\Phi^s\left(\Delta\right)} \right\} \left(\frac{\mathrm{Z}\left(\Delta\right)}{\varepsilon_{10}^* + \Phi^s\left(\Delta\right)} + 1 \right)$$
$$- \left(\alpha_{10} + \alpha_{11}\right)$$
$$0 = \alpha_{01} \left[\left(1 - \chi_{01} \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}}\right) \mathrm{Z}\left(\Delta\right) - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \chi_{01} \frac{\varepsilon_{10}^* + \Phi^s\left(\Delta\right)}{\varepsilon_{11}^* + \Phi^s\left(\Delta\right)} \lambda \Phi^s\left(\Delta\right) \right]$$
$$+ \alpha_{11} \left\{ G\left(\varepsilon_{11}^*\right) \left[1 - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \left(1 - \chi_{11}\right)\right] \right.$$
$$\left. - \left[1 - G\left(\varepsilon_{11}^*\right)\right] \frac{\lambda \Phi^s\left(\Delta\right)}{\varepsilon_{11}^* + \left(1-\lambda\right)\Phi^s\left(\Delta\right)} \right\} \left[\mathrm{Z}\left(\Delta\right) + \varepsilon_{10}^* + \Phi^s\left(\Delta\right)\right]$$

where $\chi_{01}, \chi_{11} \in [0,1]$, and

$$i^p = \left(\alpha_{01} + \alpha_{11}\right) \theta \frac{\varepsilon_{11}^* - \varepsilon_{10}^*}{\varepsilon_{10}^* + \Phi^s\left(\Delta\right)}$$
$$+ \left[\alpha_{10} + \alpha_{11}\left(1-\theta\right)\right] \frac{1}{\varepsilon_{10}^* + \Phi^s\left(\Delta\right)} \int_{\varepsilon_{10}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{10}^*\right) dG(\varepsilon)$$
$$+ \alpha_{11} \theta \frac{\varepsilon_{11}^* + \Phi^s\left(\Delta\right)}{\varepsilon_{10}^* + \Phi^s\left(\Delta\right)} \frac{1}{\varepsilon_{11}^* + \left(1-\lambda\right)\Phi^s\left(\Delta\right)} \int_{\varepsilon_{11}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11}^*\right) dG(\varepsilon)$$
$$\frac{r + \delta - g + g\delta\Delta}{\left(1 + g\Delta\right)\left(1 - \delta\Delta\right)} \Phi^s\left(\Delta\right)\Delta = \bar{\varepsilon} + \left[\alpha_{10} + \alpha_{11}\left(1-\theta\right)\right] \int_{\varepsilon_L}^{\varepsilon_{10}^*} \left(\varepsilon_{10}^* - \varepsilon\right) dG\left(\varepsilon\right)$$
$$+ \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} \left(\varepsilon_{11}^* - \varepsilon\right) dG\left(\varepsilon\right) \right.$$
$$\left. + \frac{\lambda \Phi^s\left(\Delta\right)}{\varepsilon_{11}^* + \left(1-\lambda\right)\Phi^s\left(\Delta\right)} \int_{\varepsilon_{11}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11}^*\right) dG\left(\varepsilon\right) \right].$$

These conditions can be rewritten as

$$0 = \left\{ \alpha_{10} \left[1 - G\left(\varepsilon_{10}^*\right)\right] + \alpha_{11} \left[1 - G\left(\varepsilon_{11}^*\right)\right] \frac{\varepsilon_{11}^*\Delta + \Phi^s\left(\Delta\right)\Delta}{\varepsilon_{11}^*\Delta + \left(1-\lambda\right)\Phi^s\left(\Delta\right)\Delta} \right\} \left(\frac{\mathrm{Z}\left(\Delta\right)\Delta}{\varepsilon_{10}^*\Delta + \Phi^s\left(\Delta\right)\Delta} + 1 \right)$$
$$- \left(\alpha_{10} + \alpha_{11}\right)$$
$$0 = \alpha_{01} \left[\left(1 - \chi_{01} \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}}\right) \mathrm{Z}\left(\Delta\right)\Delta - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \chi_{01} \frac{\varepsilon_{10}^*\Delta + \Phi^s\left(\Delta\right)\Delta}{\varepsilon_{11}^*\Delta + \Phi^s\left(\Delta\right)\Delta} \lambda \Phi^s\left(\Delta\right)\Delta \right]$$
$$+ \alpha_{11} \left\{ G\left(\varepsilon_{11}^*\right) \left[1 - \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \left(1 - \chi_{11}\right)\right] \right.$$
$$\left. - \left[1 - G\left(\varepsilon_{11}^*\right)\right] \frac{\lambda \Phi^s\left(\Delta\right)\Delta}{\varepsilon_{11}^*\Delta + \left(1-\lambda\right)\Phi^s\left(\Delta\right)\Delta} \right\} \left[\mathrm{Z}\left(\Delta\right)\Delta + \varepsilon_{10}^*\Delta + \Phi^s\left(\Delta\right)\Delta\right]$$

where $\chi_{01}, \chi_{11} \in [0,1]$, and

$$\begin{aligned}
\frac{i^p}{\Delta} &= (\alpha_{01} + \alpha_{11})\,\theta \frac{\varepsilon_{11}^* - \varepsilon_{10}^*}{\varepsilon_{10}^*\Delta + \Phi^s(\Delta)\Delta} \\
&\quad + [\alpha_{10} + \alpha_{11}(1-\theta)] \frac{1}{\varepsilon_{10}^*\Delta + \Phi^s(\Delta)\Delta} \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) \\
&\quad + \alpha_{11}\theta \frac{\varepsilon_{11}^*\Delta + \Phi^s(\Delta)\Delta}{\varepsilon_{10}^*\Delta + \Phi^s(\Delta)\Delta} \frac{1}{\varepsilon_{11}^*\Delta + (1-\lambda)\Phi^s(\Delta)\Delta} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \\
\frac{r+\delta-g+g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\Delta &= \bar{\varepsilon} + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) \\
&\quad + \alpha_{11}\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon) \\
&\quad + \frac{\lambda \Phi^s(\Delta)\Delta}{\varepsilon_{11}^*\Delta + (1-\lambda)\Phi^s(\Delta)\Delta} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \Bigg].
\end{aligned}$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ■

Corollary 7 *Assume $G(\varepsilon) = G_L \mathbb{I}_{\{\varepsilon_L \leq \varepsilon < \varepsilon_H\}} + \mathbb{I}_{\{\varepsilon_H \leq \varepsilon\}}$. Consider the limiting economy (as $\Delta \to 0$) where brokers do not hold assets overnight. A recursive nonmonetary equilibrium is a pair $(\varepsilon^n, \varphi^n)$ that satisfies*

$$1 = \frac{1}{1-\lambda} \sum_{k \in \{L,H\}} G_k \chi(\varepsilon^n, \varepsilon_k) \tag{167}$$

$$\varphi^n = \bar{\varepsilon} + \alpha_{11}\theta \sum_{k \in \{L,H\}} G_k (\varepsilon^n - \varepsilon_k) \left[\mathbb{I}_{\{\varepsilon_k < \varepsilon^n\}} - \frac{\lambda}{1-\lambda} \mathbb{I}_{\{\varepsilon^n < \varepsilon_k\}} \right], \tag{168}$$

together with a mixed indicator function $\chi(\varepsilon^n, \varepsilon_k) \in [0,1]$ for $\varepsilon_k \in \{\varepsilon_L, \varepsilon_H\}$.

Corollary 8 *Assume $G(\varepsilon) = G_L \mathbb{I}_{\{\varepsilon_L \leq \varepsilon < \varepsilon_H\}} + \mathbb{I}_{\{\varepsilon_H \leq \varepsilon\}}$. Consider the limiting economy (as $\Delta \to 0$) where brokers do not hold assets overnight. A recursive monetary equilibrium (with credit) is a vector $(\varepsilon_{10}^*, \varepsilon_{11}^*, \varphi, \mathcal{Z})$ that satisfies the market-clearing conditions for equity and* *bonds*

$$0=\left(\frac{\mathcal{Z}}{\varphi}+1\right)\left\{\alpha_{10}\left[G_L\chi_{10}^{L}\left(\varepsilon_{10}^{*},\varepsilon_L\right)+G_H\chi_{10}^{H}\left(\varepsilon_{10}^{*},\varepsilon_H\right)\right]\right.$$
$$\left.+\,\alpha_{11}\left[G_L\chi_{11}^{sL}\left(\varepsilon_{11}^{*},\varepsilon_L\right)+G_H\chi_{11}^{sH}\left(\varepsilon_{11}^{*},\varepsilon_H\right)\right]\frac{1}{1-\lambda}\right\}$$
$$-\left(\alpha_{10}+\alpha_{11}\right)$$

$$0=\alpha_{01}\left[\left(1-\chi_{01}\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\right)\frac{\mathcal{Z}}{\varphi}-\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\chi_{01}\lambda\right]$$
$$+\,\alpha_{11}\left(\frac{\mathcal{Z}}{\varphi}+1\right)G_L\left\{\left[1-\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\left(1-\chi_{11}^{bL}\right)\right]\mathbb{I}_{\left\{\varepsilon_L<\varepsilon_{11}^{*}\right\}}-\frac{\lambda}{1-\lambda}\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_L\right\}}\right.$$
$$\left.+\left[\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_{11}^{*}\right\}}\left(1-\chi_{11}^{sL}\right)-\chi_{11}^{sL}\frac{\lambda}{1-\lambda}\right]\mathbb{I}_{\left\{\varepsilon_L=\varepsilon_{11}^{*}\right\}}\right\}$$
$$+\,\alpha_{11}\left(\frac{\mathcal{Z}}{\varphi}+1\right)G_H\left\{\left[1-\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\left(1-\chi_{11}^{bH}\right)\right]\mathbb{I}_{\left\{\varepsilon_H<\varepsilon_{11}^{*}\right\}}-\frac{\lambda}{1-\lambda}\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_H\right\}}\right.$$
$$\left.+\left[\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_{11}^{*}\right\}}\left(1-\chi_{11}^{sH}\right)-\chi_{11}^{sH}\frac{\lambda}{1-\lambda}\right]\mathbb{I}_{\left\{\varepsilon_H=\varepsilon_{11}^{*}\right\}}\right\}$$

and the Euler equations for equity and money

$$\varphi=\bar{\varepsilon}+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\left[G_L\mathbb{I}_{\left\{\varepsilon_L\le\varepsilon_{10}^{*}\right\}}\left(\varepsilon_{10}^{*}-\varepsilon_L\right)+G_H\mathbb{I}_{\left\{\varepsilon_H\le\varepsilon_{10}^{*}\right\}}\left(\varepsilon_{10}^{*}-\varepsilon_H\right)\right]$$
$$+\,\alpha_{11}\theta\left\{G_L\left(\varepsilon_{11}^{*}-\varepsilon_L\right)\left[\mathbb{I}_{\left\{\varepsilon_L\le\varepsilon_{11}^{*}\right\}}-\frac{\lambda}{1-\lambda}\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_L\right\}}\right]\right.$$
$$\left.+\,G_H\left(\varepsilon_{11}^{*}-\varepsilon_H\right)\left[\mathbb{I}_{\left\{\varepsilon_H\le\varepsilon_{11}^{*}\right\}}-\frac{\lambda}{1-\lambda}\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_H\right\}}\right]\right\}$$

$$\iota\varphi=\left(\alpha_{01}+\alpha_{11}\right)\theta\left(\varepsilon_{11}^{*}-\varepsilon_{10}^{*}\right)$$
$$+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\left[G_L\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_L\right\}}\left(\varepsilon_L-\varepsilon_{10}^{*}\right)+G_H\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_H\right\}}\left(\varepsilon_H-\varepsilon_{10}^{*}\right)\right]$$
$$+\,\alpha_{11}\theta\frac{1}{1-\lambda}\left[G_L\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_L\right\}}\left(\varepsilon_L-\varepsilon_{11}^{*}\right)+G_H\mathbb{I}_{\left\{\varepsilon_{11}^{*}<\varepsilon_H\right\}}\left(\varepsilon_H-\varepsilon_{11}^{*}\right)\right],$$

where $\chi_{01},\chi_{11}^{bL},\chi_{11}^{sL},\chi_{11}^{bH},\chi_{11}^{sH}\in[0,1]$, *and* $\chi_{10}^{L}(\cdot,\cdot)$, $\chi_{10}^{H}(\cdot,\cdot)$, $\chi_{11}^{sL}(\cdot,\cdot)$ *and* $\chi_{11}^{sH}(\cdot,\cdot)$ *are "mixed indicator functions".*

Since the distribution $G(\varepsilon)=G_L\mathbb{I}_{\{\varepsilon_L\le\varepsilon<\varepsilon_H\}}+\mathbb{I}_{\{\varepsilon_H\le\varepsilon\}}$ consists of two mass points, an equilibrium may involve sets of agents with strictly positive measure who are indifferent between a pair of assets. For example, if $\varepsilon_{10}^{*}=\varepsilon_{11}^{*}$, then the nominal interest rate on inside bonds is zero, and type 01 investors are indifferent between holding bonds or money. For this reason, in general we introduce the variable χ_{01} that represents the fraction of beginning-of-period financial wealth that an investor of type 01 chooses to hold in the form of bonds. Similarly, for $j \in \{L, H\}$, $\chi_{10}^{j}(\varepsilon_{10}^{*}, \varepsilon_j)$ is the fraction of beginning-of-period financial wealth that an investor of type 10 and valuation ε_j chooses to hold in the form equity by the end of the OTC round, $\chi_{11}^{sj}(\varepsilon_{11}^{*}, \varepsilon_j)$ is the fraction of beginning-of-period financial wealth that an investor of type 11 and valuation ε_j chooses to hold in the form equity by the end of the OTC round, $\chi_{11}^{bj}(\varepsilon_{11}^{*}, \varepsilon_j)$ is the fraction of beginning-of-period financial wealth that an investor of type 11 and valuation ε_j chooses to hold in the form bonds by the end of the OTC round, $\chi_{11}^{sj} \equiv \chi_{11}^{sj}(\varepsilon_j, \varepsilon_j)$, and $\chi_{11}^{bj} \equiv \chi_{11}^{bj}(\varepsilon_j, \varepsilon_j)$.

A.6.2 Existence of equilibrium

Proof of Proposition 1. The conditions (41) and (42) in the statement of the proposition are the equilibrium conditions derived in Lemma 12. Clearly for any $\lambda \in [0, 1]$ there is a unique ε^n that satisfies (42), and given ε^n, the normalized equity price φ is given by (41). ■

Lemma 14 *In a RNE,*

$$\frac{d\varepsilon^n}{d\lambda} = \frac{1}{G'(\varepsilon^n)} > 0$$

$$\frac{d\varphi^n}{d\lambda} = \alpha_{11}\theta \frac{1}{(1-\lambda)^2} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) > 0.$$

Proof. The first result is obtained by implicitly differentiating (42). For the second result, differentiate (41):

$$\begin{aligned}
\frac{d}{d\lambda}\varphi^n &= \alpha_{11}\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \\
&= \alpha_{11}\theta \left[G(\varepsilon^n) \frac{d\varepsilon^n}{d\lambda} - \frac{\lambda}{1-\lambda} [1 - G(\varepsilon^n)] \frac{d\varepsilon^n}{d\lambda} + \frac{1}{(1-\lambda)^2} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \\
&= \alpha_{11}\theta \frac{1}{(1-\lambda)^2} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon).
\end{aligned}$$

■

Proof of Proposition 2. Set $\alpha_{01} = 0$ in the equilibrium conditions reported in Lemma 13 to obtain

$$0 = \left\{ \alpha_{10} [1 - G(\varepsilon_{10}^{*})] + \alpha_{11} [1 - G(\varepsilon_{11}^{*})] \frac{1}{1-\lambda} \right\} \left(\frac{\mathcal{Z}}{\varphi} + 1 \right) - (\alpha_{10} + \alpha_{11}) \tag{169}$$

$$0 = G\left(\varepsilon_{11}^{*}\right)\left[1-\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\left(1-\chi_{11}\right)\right]-\left[1-G\left(\varepsilon_{11}^{*}\right)\right]\frac{\lambda}{1-\lambda} \tag{170}$$

$$\begin{aligned}\iota\varphi &= \alpha_{11}\theta\left(\varepsilon_{11}^{*}-\varepsilon_{10}^{*}\right)+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{10}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{10}^{*}\right)dG(\varepsilon)\\ &+\alpha_{11}\theta\frac{1}{1-\lambda}\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG(\varepsilon)\end{aligned} \tag{171}$$

$$\begin{aligned}\varphi &= \bar{\varepsilon}+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{L}}^{\varepsilon_{10}^{*}}\left(\varepsilon_{10}^{*}-\varepsilon\right)dG\left(\varepsilon\right)\\ &+\alpha_{11}\theta\left[\int_{\varepsilon_{L}}^{\varepsilon_{11}^{*}}\left(\varepsilon_{11}^{*}-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda}{1-\lambda}\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG\left(\varepsilon\right)\right]\end{aligned} \tag{172}$$

where $\chi_{11}\in[0,1]$. These are four equations in four unknowns. The unknowns are $(\varepsilon_{10}^{*},\varepsilon_{11}^{*},\varphi,\mathcal{Z})$ if $\varepsilon_{10}^{*}<\varepsilon_{11}^{*}$, or $(\varepsilon^{*},\chi_{11},\varphi,\mathcal{Z})$ if $\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\equiv\varepsilon^{*}$ (recall (20) and Lemma 10 imply $\varepsilon_{10}^{*}\leq\varepsilon_{11}^{*}$ in a monetary equilibrium with credit). We consider each case in turn.

(i) Suppose $\varepsilon_{10}^{*}<\varepsilon_{11}^{*}$. In this case, (170) implies $\varepsilon_{11}^{*}=\varepsilon^{n}$, where $\varepsilon^{n}\in[\varepsilon_{L},\varepsilon_{H}]$ is the unique solution to $G\left(\varepsilon^{n}\right)=\lambda$. Combined, conditions (171) and (172) imply a single equation in the unknown ε_{10}^{*} that can be written as $T\left(\varepsilon_{10}^{*}\right)=0$, where

$$\begin{aligned}T\left(x\right) &\equiv \alpha_{11}\theta\left(\varepsilon^{n}-x\right)+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{x}^{\varepsilon_{H}}\left(\varepsilon-x\right)dG(\varepsilon)+\alpha_{11}\theta\frac{1}{1-\lambda}\int_{\varepsilon^{n}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{n}\right)dG(\varepsilon)\\ &-\iota\left\{\bar{\varepsilon}+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{L}}^{x}\left(x-\varepsilon\right)dG\left(\varepsilon\right)\right.\\ &\left.+\alpha_{11}\theta\left[\int_{\varepsilon_{L}}^{\varepsilon^{n}}\left(\varepsilon^{n}-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^{n}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{n}\right)dG\left(\varepsilon\right)\right]\right\}.\end{aligned}$$

Differentiate T and evaluate the derivative at $x=\varepsilon_{10}^{*}$ to obtain

$$T'\left(\varepsilon_{10}^{*}\right)=-\left\{\alpha_{11}\theta+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\left\{\left[1-G\left(\varepsilon_{10}^{*}\right)\right]+\iota G\left(\varepsilon_{10}^{*}\right)\right\}\right\}<0.$$

Hence if there is a ε_{10}^{*} that satisfies $T\left(\varepsilon_{10}^{*}\right)=0$, it is unique. Notice that

$$\begin{aligned}T\left(\varepsilon_{L}\right) &= \alpha_{11}\theta\left(\varepsilon^{n}-\varepsilon_{L}\right)+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\left(\bar{\varepsilon}-\varepsilon_{L}\right)+\alpha_{11}\theta\frac{1}{1-\lambda}\int_{\varepsilon^{n}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{n}\right)dG(\varepsilon)\\ &-\iota\left\{\bar{\varepsilon}+\alpha_{11}\theta\left[\int_{\varepsilon_{L}}^{\varepsilon^{n}}\left(\varepsilon^{n}-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^{n}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{n}\right)dG\left(\varepsilon\right)\right]\right\},\end{aligned}$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\iota}(\lambda)$, where $\bar{\iota}(\lambda)$ is defined in the statement of the proposition. Also,

$$\begin{aligned} T(\varepsilon^n) = & \left[\alpha_{10} + \alpha_{11}\left(1 + \theta\frac{\lambda}{1-\lambda}\right)\right]\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\ & - \iota\left\{\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right\}, \end{aligned}$$

so $T(\varepsilon^n) < 0$ if and only if $\hat{\iota}(\lambda) < \iota$. Thus if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, there exists a unique ε_{10}^* that satisfies $T(\varepsilon_{10}^*) = 0$, and $\varepsilon_{10}^* \in (\varepsilon_L, \varepsilon^n)$. Given ε_{10}^* and ε_{11}^*, φ is given by (172). Finally, given ε_{10}^*, ε_{11}^*, and φ, $\mathcal{Z}$ is given by (169), which can be written as (46). From this expression, it is clear that $0 < \mathcal{Z} \Leftrightarrow \alpha_{10} > 0$ and $\varepsilon_L < \varepsilon_{10}^*$ (and the latter condition is implied by $\iota < \bar{\iota}(\lambda)$).

(*ii*) Suppose $\varepsilon_{10}^* = \varepsilon_{11}^* \equiv \varepsilon^*$. In this case, (169)-(172) become

$$\alpha_{10} + \alpha_{11} = [1 - G(\varepsilon^*)]\left(\alpha_{10} + \alpha_{11}\frac{1}{1-\lambda}\right)\left(\frac{\mathcal{Z}}{\varphi} + 1\right) \tag{173}$$

$$\chi_{11} = \frac{\lambda}{1-\lambda}\frac{1 - G(\varepsilon^*)}{G(\varepsilon^*)} \tag{174}$$

$$\iota\varphi = \left[\alpha_{10} + \alpha_{11}\left(1 - \theta + \theta\frac{1}{1-\lambda}\right)\right]\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\, dG(\varepsilon) \tag{175}$$

$$\varphi = \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta\frac{\lambda}{1-\lambda}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\, dG(\varepsilon). \tag{176}$$

Conditions (175) and (176) imply a single equation in the unknown ε^* that can be written as $\mathcal{T}(\varepsilon^*) = 0$, where

$$\begin{aligned} \mathcal{T}(\varepsilon^*) \equiv & \left\{\alpha_{10} + \alpha_{11}\left[1 + (1-\iota)\,\theta\frac{\lambda}{1-\lambda}\right]\right\}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\ & - \iota\left[\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\, dG(\varepsilon)\right]. \end{aligned}$$

Differentiate $\mathcal{T}$ and evaluate the derivative at the ε^* that solves $\mathcal{T}(\varepsilon^*) = 0$ to obtain

$$\mathcal{T}'(\varepsilon^*) = -\iota\left\{\frac{\bar{\varepsilon} + (\alpha_{10}+\alpha_{11})\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)dG(\varepsilon)}{\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)dG(\varepsilon)}[1 - G(\varepsilon^*)] + (\alpha_{10} + \alpha_{11})\, G(\varepsilon^*)\right\} \leq 0,$$

with "=" only if $\iota = 0$. Hence if there is a ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, it is unique. Notice that

$$\mathcal{T}(\varepsilon_H) = -\iota\left[\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})\right],$$

so $\mathcal{T}(\varepsilon_H) < 0$ if and only if $0 < \iota$. Also,

$$\mathcal{T}(\varepsilon^n) = \left\{\alpha_{10} + \alpha_{11}\left[1 + (1-\iota)\,\theta\frac{\lambda}{1-\lambda}\right]\right\}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)$$
$$- \iota\left[\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon)\right],$$

so $0 \leq \mathcal{T}(\varepsilon^n)$ if and only if $\iota \leq \hat{\imath}(\lambda)$. Thus if $0 < \iota \leq \hat{\imath}(\lambda)$, there exists a unique ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, and $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\imath}(\lambda)$). Given ε^*, $\chi_{11} \in [0,1]$ is given by (174) and φ is given by (176). Finally, given ε^* and φ, (173) implies $\mathcal{Z}$. ■

Lemma 15 *The real asset price in the RME is higher than the real asset price in the RNE, i.e.,*

(i) If $\hat{\imath}(\lambda) < \iota < \bar{\imath}(\lambda)$, *then*

$$0 < [\alpha_{10} + \alpha_{11}(1-\theta)]\int_{\varepsilon_L}^{\varepsilon_{10}^*}(\varepsilon_{10}^* - \varepsilon)\,dG(\varepsilon) \leq \varphi - \varphi^n. \tag{177}$$

(ii) If $0 < \iota \leq \hat{\imath}(\lambda)$, *then*

$$0 < [\alpha_{10} + \alpha_{11}(1-\theta)]\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\,dG(\varepsilon) \leq \varphi - \varphi^n. \tag{178}$$

Moreover, in any RME, $\varphi \leq \psi \equiv \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})$, *with "=" only if* $\iota = 0$.

Proof. (i) If $\hat{\imath}(\lambda) < \iota < \bar{\imath}(\lambda)$, (177) is immediate from (45). (ii) If $0 < \iota \leq \hat{\imath}(\lambda)$, use (41) and the expression for φ in part (ii) of Proposition 2 to write

$$\varphi - \varphi^n = \alpha_{10}\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\,dG(\varepsilon)$$
$$+ \alpha_{11}\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^* - \varepsilon)\,dG(\varepsilon) + \alpha_{11}\theta\frac{\lambda}{1-\lambda}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon)$$
$$- \alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right].$$

Define

$$\Upsilon(x) \equiv \int_{\varepsilon_L}^{x}(x - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{x}^{\varepsilon_H}(\varepsilon - x)\,dG(\varepsilon) \tag{179}$$

and notice that for all $x \in [\varepsilon^n, \varepsilon_H]$,

$$\Upsilon'(x) = G(x) - \frac{\lambda}{1-\lambda}[1 - G(x)] \geq 0, \text{ with " = " only if } x = \varepsilon^n. \tag{180}$$

Thus, since $0 < \iota \leq \hat{\iota}(\lambda)$ implies $\varepsilon^n \leq \varepsilon^*$, we have

$$\begin{aligned}\varphi - \varphi^n \geq{}& \alpha_{10} \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\ &+ \alpha_{11} \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\ &- \alpha_{11}\theta \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \right],\end{aligned}$$

which implies (178).

To show that $\varphi \leq \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})$, we again consider two cases. First, suppose $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$. In this case,

$$\begin{aligned}\varphi - \psi ={}& \alpha_{10} \left[\int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\ &+ \alpha_{11} \left[\theta \Upsilon(\varepsilon^n) + (1-\theta) \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\ &< 0.\end{aligned}$$

Second, suppose $0 < \iota \leq \hat{\iota}(\lambda)$. In this case,

$$\begin{aligned}\varphi - \psi ={}& \alpha_{10} \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\ &+ \alpha_{11} \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\ \leq{}& \alpha_{10} \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] + \alpha_{11} \left[\Upsilon(\varepsilon^*) - (\varepsilon_H - \bar{\varepsilon}) \right] \\ \leq{}& 0.\end{aligned}$$

To conclude, notice the last inequality is strict unless $\iota \to 0$, which implies $\varepsilon^* \to \varepsilon_H$, and therefore $\varphi \to \psi$. ■

Proof of Proposition 3. Part (i) is an immediate corollary of Lemma 15. For part (ii), we consider two cases in turn.

If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then

$$\frac{d\varepsilon_{10}^*}{d\iota} = -\frac{\frac{\partial T(\varepsilon_{10}^*)}{\partial \iota}}{T'(\varepsilon_{10}^*)} = -\frac{-\varphi}{-\left\{ \alpha_{11}\theta + [\alpha_{10} + \alpha_{11}(1-\theta)] \left\{ [1 - G(\varepsilon_{10}^*)] + \iota G(\varepsilon_{10}^*) \right\} \right\}} < 0,$$

where $T(\cdot)$ is the equilibrium map defined in part (i) of the proof of Proposition 2. Then, from (45),

$$\frac{d\varphi}{d\iota} = [\alpha_{10} + \alpha_{11}(1-\theta)]\, G(\varepsilon_{10}^*)\, \frac{d\varepsilon_{10}^*}{d\iota} < 0.$$

If $0 < \iota \le \hat{\iota}(\lambda)$, then

$$\frac{d\varepsilon^*}{d\iota} = -\frac{\frac{\partial \mathcal{T}(\varepsilon^*)}{\partial \iota}}{\mathcal{T}'(\varepsilon^*)} = -\frac{-\varphi}{-\iota\left\{\frac{\bar{\varepsilon}+(\alpha_{10}+\alpha_{11})\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)dG(\varepsilon)}{\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)dG(\varepsilon)}[1-G(\varepsilon^*)] + (\alpha_{10}+\alpha_{11})\,G(\varepsilon^*)\right\}} < 0,$$

where $\mathcal{T}(\cdot)$ is the equilibrium map defined in part (ii) of the proof of Proposition 2. Then, differentiating the expression for φ in part (ii) of the statement of Proposition 2,

$$\frac{d\varphi}{d\iota} = \left[(\alpha_{10}+\alpha_{11})\, G(\varepsilon^*) - \alpha_{11}\theta\frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)]\right]\frac{d\varepsilon^*}{d\iota}.$$

Notice that

$$\begin{aligned} 0 &= (\alpha_{10}+\alpha_{11})\left[G(\varepsilon^n) - \frac{\lambda}{1-\lambda}[1-G(\varepsilon^n)]\right] \\ &\le (\alpha_{10}+\alpha_{11})\left[G(\varepsilon^*) - \frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)]\right] \\ &< (\alpha_{10}+\alpha_{11})\, G(\varepsilon^*) - \alpha_{11}\theta\frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)], \end{aligned}$$

where the first inequality follows because $G(x) - \frac{\lambda}{1-\lambda}[1-G(x)]$ is increasing in x, and $\varepsilon^n \le \varepsilon^*$ for all $0 < \iota \le \hat{\iota}(\lambda)$. Hence, $d\varphi/d\iota < 0$. ■

Proof of Proposition 4. A nonmonetary equilibrium of this economy is a pair $(\varepsilon^n, \varphi^n)$ that satisfies (167) and (168), together with a function $\chi(\varepsilon^n, \varepsilon_k) \in [0,1]$ for $\varepsilon_k \in \{\varepsilon_L, \varepsilon_H\}$. For any $\varepsilon \in \mathbb{R}$, define

$$\begin{aligned} 1 - \tilde{G}(\varepsilon) &\equiv \sum_{j\in\{L,H\}} G_j \chi(\varepsilon, \varepsilon_j) \\ &= \mathbb{I}_{\{\varepsilon<\varepsilon_L\}} + (G_L\tilde{\chi}_L + G_H)\,\mathbb{I}_{\{\varepsilon=\varepsilon_L\}} + G_H\mathbb{I}_{\{\varepsilon_L<\varepsilon<\varepsilon_H\}} + G_H\tilde{\chi}_H\mathbb{I}_{\{\varepsilon=\varepsilon_H\}}, \end{aligned}$$

where $\tilde{\chi}_L, \tilde{\chi}_H \in [0,1]$. For any $\varepsilon \in \mathbb{R}$, define the mapping $T(\varepsilon, \chi_L, \chi_H) \equiv \lambda - \tilde{G}(\varepsilon)$, with $\chi_k \equiv 1 - \tilde{\chi}_k$ for $k \in \{L, H\}$. More explicitly,

$$T(\varepsilon, \chi_L, \chi_H) = \begin{cases} \lambda & \text{if } \varepsilon < \varepsilon_L \\ \lambda - G_L\chi_L & \text{if } \varepsilon = \varepsilon_L \\ \lambda - G_L & \text{if } \varepsilon_L < \varepsilon < \varepsilon_H \\ \lambda - G_L - G_H\chi_H & \text{if } \varepsilon = \varepsilon_H. \end{cases}$$

An equilibrium can now be described by a triple $(\varepsilon^n, \chi_L, \chi_H)$, with $\chi_L, \chi_H \in [0,1]$ that satisfies $T(\varepsilon^n, \chi_L, \chi_H) = 0$. Given ε^n, φ^n is obtained from (168). There are three possibilities: either $\lambda < G_L$, or $G_L < \lambda$, or $\lambda = G_L$. First, if $\lambda < G_L$, then the equilibrium is $(\varepsilon^n, \chi_L, \chi_H) = (\varepsilon_L, \chi_L^*, \chi_H)$ and $\chi_H \in [0,1]$, where

$$\chi_L^* = \frac{\lambda}{G_L}.$$

To see this, verify that $T(\varepsilon', \chi_L, \chi_H) < T(\varepsilon_L, \chi_L^*, \chi_H) = 0 < T(\varepsilon'', \chi_L, \chi_H)$ for all $\varepsilon'' < \varepsilon_L$, all $\varepsilon' > \varepsilon_L$, and any $\chi_L, \chi_H \in [0,1]$. Second, if $G_L < \lambda$, then the equilibrium is $(\varepsilon^n, \chi_L, \chi_H) = (\varepsilon_H, \chi_L, \chi_H^*)$ and $\chi_L \in [0,1]$, where

$$\chi_H^* = \frac{\lambda - G_L}{G_H}.$$

To see this, verify that $T(\varepsilon_H, \chi_L, \chi_H^*) = 0 < T(\varepsilon', \chi_L, \chi_H)$ for all $\varepsilon' < \varepsilon_H$, and any $\chi_L, \chi_H \in [0,1]$. Finally, if $\lambda = G_L$, then an equilibrium is any $(\varepsilon^n, \chi_L, \chi_H)$ with $\varepsilon^n \in (\varepsilon_L, \varepsilon_H)$ and $\chi_L, \chi_H \in [0,1]$, as well as $(\varepsilon^n, \chi_L, \chi_H) = (\varepsilon_L, 1, \chi_H)$ and $\chi_H \in [0,1]$, or $(\varepsilon^n, \chi_L, \chi_H) = (\varepsilon_H, \chi_L, 0)$ and $\chi_L \in [0,1]$. ■

Proof of Proposition 5. The equilibrium conditions are given in Corollary 8. An equilibrium can take one of seven forms: (A1) $\varepsilon_L = \varepsilon_{10}^* = \varepsilon_{11}^* < \varepsilon_H$, (A2) $\varepsilon_L < \varepsilon_{10}^* = \varepsilon_{11}^* < \varepsilon_H$, (A3) $\varepsilon_L < \varepsilon_{10}^* = \varepsilon_{11}^* = \varepsilon_H$, (B1) $\varepsilon_L = \varepsilon_{10}^* < \varepsilon_{11}^* < \varepsilon_H$, (B2) $\varepsilon_L = \varepsilon_{10}^* < \varepsilon_{11}^* = \varepsilon_H$, (B3) $\varepsilon_L < \varepsilon_{10}^* < \varepsilon_{11}^* < \varepsilon_H$, (B4) $\varepsilon_L < \varepsilon_{10}^* < \varepsilon_{11}^* = \varepsilon_H$. We consider each in turn.

(A1) If $\varepsilon_L = \varepsilon_{10}^* = \varepsilon_{11}^* < \varepsilon_H$, the Euler equations for money and equity imply

$$\varphi = \bar{\varepsilon} + \alpha_{11}\theta\pi_H \frac{\lambda}{1-\lambda}(\varepsilon_H - \varepsilon_L)$$

and also $\iota = \hat{\iota}(\lambda)$, so this case is of measure zero in the space of parameters.

(A2) If $\varepsilon_L < \varepsilon_{10}^* = \varepsilon_{11}^* \equiv \varepsilon^* < \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left(\alpha_{10} + \alpha_{11}\frac{1}{1-\lambda}\right) G_H - (\alpha_{10} + \alpha_{11}) A^s \tag{181}$$

$$0 = \alpha_{01}\left[(1-\chi_{01})\frac{\mathcal{Z}}{\varphi} - \chi_{01}\lambda\right] + \left(\alpha_{11} G_L \chi_{11}^{bL} - \alpha_{11} G_H \frac{\lambda}{1-\lambda}\right)\left(\frac{\mathcal{Z}}{\varphi} + 1\right) \tag{182}$$

$$\iota\varphi = \left\{\alpha_{10} + \alpha_{11}\left[1 + \theta\left(\frac{\lambda}{1-\lambda}\right)\right]\right\} G_H(\varepsilon_H - \varepsilon^*) \tag{183}$$

$$\varphi = \bar{\varepsilon} + [\alpha_{10} + \alpha_{11}(1-\theta)] G_L(\varepsilon^* - \varepsilon_L) + \alpha_{11}\theta\left[G_L(\varepsilon^* - \varepsilon_L) + G_H(\varepsilon_H - \varepsilon^*)\frac{\lambda}{1-\lambda}\right]. \tag{184}$$

Conditions (183) and (184) imply (53) and (54), and condition (181) implies (55). Condition (182) (together with (181)) implies that χ_{01} and χ_{11}^{bL} must satisfy $\chi_{11}^{bL} = \vartheta(\chi_{01})$, where

$$\vartheta(\chi_{01}) \equiv \frac{\alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11}) G_H \frac{\lambda}{1-\lambda} - \alpha_{01}(\alpha_{10}+\alpha_{11}) G_L}{\alpha_{11}(\alpha_{10}+\alpha_{11}) G_L} + \frac{\alpha_{01}\{\alpha_{10}[1-G_H(1-\lambda)]+\alpha_{11}G_L\}}{\alpha_{11}(\alpha_{10}+\alpha_{11}) G_L}\chi_{01}.$$

Any $\chi_{01} \in [0,1]$ is consistent with equilibrium as long as $\vartheta(\chi_{01}) = \chi_{11}^{bL} \in [0,1]$. Notice that

$$\vartheta'(\chi_{01}) = \frac{\alpha_{01}\{\alpha_{10}[1-G_H(1-\lambda)]+\alpha_{11}G_L\}}{\alpha_{11}(\alpha_{10}+\alpha_{11}) G_L} \geq 0 \text{ with " = " only if } \alpha_{01}=0.$$

Hence $\vartheta(0) \leq \vartheta(\chi_{01}) = \chi_{11}^{bL} \leq \vartheta(1)$. It follows that this equilibrium configuration does not exist if either $\vartheta(1) < 0$ or $1 < \vartheta(0)$, since in this case there does not exist $\chi_{11}^{bL} = \vartheta(\chi_{01}) \in [0,1]$ for any $\chi_{01} \in [0,1]$. Equivalently, the equilibrium exists if we have both, $0 \leq \vartheta(1)$ and $\vartheta(0) \leq 1$. The condition $0 \leq \vartheta(1)$ is equivalent to

$$0 \leq \frac{\alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11}) G_H \frac{\lambda}{1-\lambda} + \alpha_{01}\alpha_{10}G_H\lambda}{\alpha_{11}(\alpha_{10}+\alpha_{11}) G_L},$$

which always holds. The condition $\vartheta(0) \leq 1$ is equivalent to parametric restriction $\lambda \leq \hat{\lambda}$. The equilibrium is then fully described by (53) and (54), and a pair $(\chi_{01}, \chi_{11}^{bL})$ such that $\chi_{11}^{bL} = f(\chi_{01})$, and $\chi_{01} \in [\underline{\chi}_{01}, \bar{\chi}_{01}] \cap [0,1]$, where $\underline{\chi}_{01}$ is the solution to $f(\underline{\chi}_{01}) = 0$ and $\bar{\chi}_{01}$ is the solution to $f(\bar{\chi}_{01}) = 1$, i.e.,

$$\underline{\chi}_{01} = \frac{\alpha_{01}(\alpha_{10}+\alpha_{11}) G_L - \alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11}) G_H \frac{\lambda}{1-\lambda}}{\alpha_{01}\{\alpha_{10}[1-G_H(1-\lambda)]+\alpha_{11}G_L\}}$$
$$\bar{\chi}_{01} = \frac{(\alpha_{11}+\alpha_{01})(\alpha_{10}+\alpha_{11}) G_L - \alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11}) G_H \frac{\lambda}{1-\lambda}}{\alpha_{01}\{\alpha_{10}[1-G_H(1-\lambda)]+\alpha_{11}G_L\}}.$$

For the equilibrium to be monetary, we need $0 < \mathcal{Z}$, or equivalently, $\lambda < \frac{(\alpha_{10}+\alpha_{11})G_L}{\alpha_{10}G_L+\alpha_{11}}$, but this condition is implied by the parametric restriction $\lambda < \hat{\lambda}$. The conjectured conditions $\varepsilon_L < \varepsilon^* < \varepsilon_H$ are equivalent to the parametric restrictions $0 < \iota < \hat{\iota}(\lambda)$. Hence this equilibrium configuration exists for $(\iota, \lambda) \in \mathcal{E}_2^m$.

(A3) If $\varepsilon_L < \varepsilon_{10}^* = \varepsilon_{11}^* = \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left(\alpha_{10} G_H \chi_{10}^H + \alpha_{11} G_H \chi_{11}^{sH} \frac{1}{1-\lambda}\right) - (\alpha_{10} + \alpha_{11}) \tag{185}$$

$$0 = \alpha_{01}\left[(1-\chi_{01})\frac{\mathcal{Z}}{\varphi} - \chi_{01}\lambda\right] + \alpha_{11}\left(G_L \chi_{11}^{bL} - G_H \chi_{11}^{sH}\frac{\lambda}{1-\lambda}\right)\left(\frac{\mathcal{Z}}{\varphi} + 1\right) \tag{186}$$

$$\iota = 0 \tag{187}$$

$$\varphi = \bar{\varepsilon} + (\alpha_{10} + \alpha_{11}) G_L (\varepsilon_H - \varepsilon_L). \tag{188}$$

The equilibrium consists of any $\left(\mathcal{Z}, \chi_{10}^H, \chi_{01}, \chi_{11}^{bL}, \chi_{11}^{sH}\right)$ that satisfies (185) and (186), as well as $\mathcal{Z} > 0$, and $\chi_{10}^H, \chi_{01}, \chi_{11}^{bL}, \chi_{11}^{sH} \in [0,1]$.

(B1) If $\varepsilon_L = \varepsilon_{10}^* < \varepsilon_{11}^* < \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left[\alpha_{10}\left(G_L \chi_{10}^L + G_H\right) + \alpha_{11} G_H \frac{1}{1-\lambda}\right] - (\alpha_{10} + \alpha_{11}) \tag{189}$$

$$0 = \alpha_{01}\frac{\mathcal{Z}}{\varphi} + \alpha_{11}\left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left(G_L - G_H \frac{\lambda}{1-\lambda}\right) \tag{190}$$

$$\begin{aligned}\iota\varphi = {} & (\alpha_{01} + \alpha_{11})\,\theta\,(\varepsilon_{11}^* - \varepsilon_L) + [\alpha_{10} + \alpha_{11}(1-\theta)]\, G_H (\varepsilon_H - \varepsilon_L) \\ & + \alpha_{11}\theta \frac{1}{1-\lambda} G_H (\varepsilon_H - \varepsilon_{11}^*)\end{aligned} \tag{191}$$

$$\varphi = \bar{\varepsilon} + \alpha_{11}\theta\left[G_L(\varepsilon_{11}^* - \varepsilon_L) + G_H(\varepsilon_H - \varepsilon_{11}^*)\frac{\lambda}{1-\lambda}\right]. \tag{192}$$

Conditions (191) and (192) imply (56) and (57). From (190), $\mathcal{Z}$ is given by (58), and from (189), χ_{10}^L is given by (59). For this configuration to be a monetary equilibrium equilibrium we need to check: (a) $0 \leq \chi_{10}^L \leq 1$, (b) $\varepsilon_L < \varepsilon_{11}^* < \varepsilon_H$, and (c) $0 < \mathcal{Z}$. The conditions in (a) are equivalent to the parametric conditions $G_L \leq \lambda \leq \hat{\lambda}$. The conditions in (b) are equivalent to the parametric conditions $\hat{\iota}(\lambda) < \iota < \bar{\iota}$. Condition (c) is implied by (a). Hence this equilibrium configuration exists for $(\iota, \lambda) \in \mathcal{E}_3^m$.

(B2) If $\varepsilon_L = \varepsilon_{10}^* < \varepsilon_{11}^* = \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left[\alpha_{10}\left(G_L \chi_{10}^L + G_H\right) + \alpha_{11} G_H \chi_{11}^{sH} \frac{1}{1-\lambda}\right] - (\alpha_{10} + \alpha_{11}) \tag{193}$$

$$0 = \alpha_{01}\frac{\mathcal{Z}}{\varphi} + \alpha_{11}\left(\frac{\mathcal{Z}}{\varphi} + 1\right)\left(1 - G_H \chi_{11}^{sH}\frac{1}{1-\lambda}\right) \tag{194}$$

$$\iota\varphi = \left\{(\alpha_{01} + \alpha_{11})\,\theta + [\alpha_{10} + \alpha_{11}(1-\theta)]\, G_H\right\}(\varepsilon_H - \varepsilon_L) \tag{195}$$

$$\varphi = \bar{\varepsilon} + \alpha_{11}\theta G_L (\varepsilon_H - \varepsilon_L). \tag{196}$$

Condition (196) gives φ. Condition (194) implies

$$\frac{\mathcal{Z}}{\varphi} = \frac{\alpha_{11}\left(G_H \chi_{11}^{sH} \frac{1}{1-\lambda} - 1\right)}{\alpha_{01} - \alpha_{11}\left(G_H \chi_{11}^{sH} \frac{1}{1-\lambda} - 1\right)}. \tag{197}$$

Given $\mathcal{Z}/\varphi$, then (193) is a single equation in the two unknowns $\chi_{10}^{L}, \chi_{11}^{sH} \in [0,1]$, namely

$$\chi_{10}^{L} = \frac{(\alpha_{01}+\alpha_{11})(\alpha_{10}+\alpha_{11}) - \alpha_{01}\alpha_{10}G_H}{\alpha_{01}\alpha_{10}G_L} - \frac{(\alpha_{01}+\alpha_{10}+\alpha_{11})\alpha_{11}G_H \frac{1}{1-\lambda}}{\alpha_{01}\alpha_{10}G_L}\chi_{11}^{sH}. \tag{198}$$

The complete characterization of equilibrium requires to find $\left(\mathcal{Z}, \chi_{10}^{L}, \chi_{11}^{sH}\right)$ with $0 < \mathcal{Z}$ and $\chi_{10}^{L}, \chi_{11}^{sH} \in [0,1]$ that satisfy (197) and (198). Notice that conditions (195) and (196) imply the parametric restriction $\iota = \bar{\iota}$, , so this case is of measure zero in the space of parameters.

(B3) If $\varepsilon_L < \varepsilon_{10}^{*} < \varepsilon_{11}^{*} < \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi}+1\right) G_H \left(\alpha_{10} + \alpha_{11}\frac{1}{1-\lambda}\right) - (\alpha_{10}+\alpha_{11}) \tag{199}$$

$$0 = \alpha_{01}\frac{\mathcal{Z}}{\varphi} + \alpha_{11}\left(\frac{\mathcal{Z}}{\varphi}+1\right)\left(G_L - G_H\frac{\lambda}{1-\lambda}\right) \tag{200}$$

$$\begin{aligned}\iota\varphi &= (\alpha_{01}+\alpha_{11})\,\theta\left(\varepsilon_{11}^{*} - \varepsilon_{10}^{*}\right) + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right] G_H\left(\varepsilon_H - \varepsilon_{10}^{*}\right) \\ &\quad + \alpha_{11}\theta\frac{1}{1-\lambda}G_H\left(\varepsilon_H - \varepsilon_{11}^{*}\right)\end{aligned} \tag{201}$$

$$\begin{aligned}\varphi &= \bar{\varepsilon} + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right] G_L\left(\varepsilon_{10}^{*} - \varepsilon_L\right) \\ &\quad + \alpha_{11}\theta\left[G_L\left(\varepsilon_{11}^{*} - \varepsilon_L\right) + G_H\left(\varepsilon_H - \varepsilon_{11}^{*}\right)\frac{\lambda}{1-\lambda}\right].\end{aligned} \tag{202}$$

Conditions (199) and (200) imply the parametric restriction $\lambda = \hat{\lambda}$, so this case is of measure zero in the space of parameters.

(B4) If $\varepsilon_L < \varepsilon_{10}^{*} < \varepsilon_{11}^{*} = \varepsilon_H$, the equilibrium conditions specialize to

$$0 = \left(\frac{\mathcal{Z}}{\varphi}+1\right) G_H \left(\alpha_{10} + \alpha_{11}\chi_{11}^{sH}\frac{1}{1-\lambda}\right) - (\alpha_{10}+\alpha_{11}) \tag{203}$$

$$0 = \alpha_{01}\frac{\mathcal{Z}}{\varphi} + \alpha_{11}\left(\frac{\mathcal{Z}}{\varphi}+1\right)\left[G_L + G_H\left(1 - \chi_{11}^{sH}\frac{1}{1-\lambda}\right)\right] \tag{204}$$

$$\iota\varphi = \left\{(\alpha_{01}+\alpha_{11})\,\theta + \left[\alpha_{10}+\alpha_{11}(1-\theta)\right] G_H\right\}\left(\varepsilon_H - \varepsilon_{10}^{*}\right) \tag{205}$$

$$\varphi = \bar{\varepsilon} + \left[\alpha_{10}+\alpha_{11}(1-\theta)\right] G_L\left(\varepsilon_{10}^{*} - \varepsilon_L\right) + \alpha_{11}\theta G_L\left(\varepsilon_H - \varepsilon_L\right). \tag{206}$$

Conditions (205) and (206) imply (50) and (51). Conditions (203) and (204) imply (52) and

$$\chi_{11}^{sH} = (1-\lambda)\frac{\alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11}) + \alpha_{10}\alpha_{01}G_L}{\alpha_{11}(\alpha_{01}+\alpha_{10}+\alpha_{11})G_H}.$$

For this configuration to be a monetary equilibrium equilibrium we need to check: (a) $0 \leq \chi_{11}^{sH} \leq 1$, (b) $\varepsilon_L < \varepsilon_{10}^{*} < \varepsilon_H$, and (c) $0 < \mathcal{Z}$. The conditions in (a) are equivalent to the parametric condition $\hat{\lambda} \leq \lambda$. The conditions in (b) are equivalent to the parametric conditions $0 < \iota < \bar{\iota}$. Condition (c) is implied by the fact that $\varphi > 0$. Hence this equilibrium configuration exists for $(\iota, \lambda) \in \mathcal{E}_1^m$. ■

A.7 Cashless limits

Proof of Proposition 6. Without loss of generality, we compute the relevant limits along a trajectory starting from any economy indexed by the (λ, ι) such that $\iota \in [\hat{\iota}(\lambda), \bar{\iota}(\lambda)]$. As $\lambda \to 1$, the mapping T defined in part (i) of the proof of Proposition 2 converges uniformly to the mapping $T_{\lambda=1}$ defined by

$$T_{\lambda=1}(x) \equiv \alpha_{11}\theta(\varepsilon_H - x) + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_x^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon)$$
$$- \iota \left\{ \bar{\varepsilon} + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta(\varepsilon_H - \bar{\varepsilon}) \right\}.$$

(This follows from the fact that $\lim_{\lambda \to 1} \frac{1}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) = \lim_{\lambda \to 1} \frac{1-G(\varepsilon^n)}{G'(\varepsilon^n)} = 0$.) Differentiate $T_{\lambda=1}$ and evaluate the derivative at $x = \varepsilon_{10}^{*}$ to obtain

$$T'_{\lambda=1}(\varepsilon_{10}^{*}) = -\left\{ \alpha_{11}\theta + [\alpha_{10} + \alpha_{11}(1-\theta)] \left\{ [1 - G(\varepsilon_{10}^{*})] + \iota G(\varepsilon_{10}^{*}) \right\} \right\} < 0.$$

Hence if there is a ε_{10}^{*} that satisfies $T(\varepsilon_{10}^{*}) = 0$, it is unique. Notice that

$$T_{\lambda=1}(\varepsilon_L) = [\bar{\varepsilon} + \alpha_{11}\theta(\varepsilon_H - \bar{\varepsilon})] [\bar{\iota}(1) - \iota],$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\iota}(1)$. Also,

$$T_{\lambda=1}(\varepsilon_H) = -\iota [\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})]$$

so $T(\varepsilon_H) < 0$ if and only if $0 < \iota$. Thus if $0 \leq \iota \leq \bar{\iota}(1)$, there exists a unique ε_{10}^{*} that satisfies $T_{\lambda=1}(\varepsilon_{10}^{*}) = 0$ (or equivalently, (64)), and $\varepsilon_{10}^{*} \in [\varepsilon_L, \varepsilon_H]$. The limiting expressions (61) and (63) are immediate from (46) and (45). Finally, (62) is the limit of the upper branch of (60). ■

Proof of Proposition 7. Without loss of generality, we compute the relevant limits along a trajectory starting from any economy indexed by the (λ, ι) such that $\iota \in [\hat{\iota}(\lambda), \bar{\iota}(\lambda)]$. From part (i) of the proof of Proposition 2, we know that $\varepsilon_{10}^{*} \to \varepsilon_L$ as $\iota \to \bar{\iota}(\lambda)$, so (45) implies (67), (46) implies (65), and the top branch of (60) implies (66). ■

Proof of Proposition 8. With $\alpha_{10} \equiv \alpha_s \alpha$ $\alpha_{11} \equiv \alpha_s (1-\alpha)$, (60) can be written as

$$\mathcal{V} = \begin{cases} \frac{\alpha_s \left\{\alpha\left[1-G(\varepsilon_{10}^*)\right]+1-\alpha\right\}\left[\alpha G(\varepsilon_{10}^*)+(1-\alpha)\lambda\right]}{\alpha G\left(\varepsilon_{10}^*\right)} & \text{if } \hat{\varsigma}(\alpha) < \iota < \bar{\varsigma}(\alpha) \\ \frac{\alpha_s\left[\alpha+(1-\alpha)\frac{1}{1-\lambda}\right]G(\varepsilon^*)[1-G(\varepsilon^*)]}{\alpha G(\varepsilon^*)+(1-\alpha)\frac{1}{1-\lambda}[G(\varepsilon^*)-\lambda]} & \text{if } 0 < \iota \leq \hat{\varsigma}(\alpha). \end{cases} \tag{207}$$

First, notice that $\hat{\varsigma}(\alpha) \leq \bar{\varsigma}(\alpha)$ for all $\alpha \in [0,1]$, with "=" only if $\lambda = 0$. Hereafter, assume $\lambda > 0$, and fix some $\iota \in (0, \bar{\iota}(0))$.

(*i*) For $\iota \in (\hat{\varsigma}(0), \bar{\varsigma}(0))$ and α small enough, part (*i*) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon_{10}^*, \varepsilon_{11}^*, \varphi, \mathcal{Z})$, where

$$\varphi = \bar{\varepsilon} + \alpha_s \Bigg\{ (1-\alpha)\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) \Bigg\} \tag{208}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon_{10}^*)}{\alpha[1 - G(\varepsilon_{10}^*)] + 1 - \alpha}\varphi \tag{209}$$

$\varepsilon_{11}^* = \varepsilon^n$, and ε_{10}^* is the unique $\varepsilon_{10}^* \in (\varepsilon_L, \varepsilon^n)$ that satisfies $\tilde{T}(\varepsilon_{10}^*; \alpha) = 0$, where for any $\varepsilon_{10}^* \in [\varepsilon_L, \varepsilon_H]$, $\tilde{T}(\cdot; \alpha)$ is a real-valued function defined by

$$\begin{aligned} \tilde{T}(\varepsilon_{10}^*; \alpha) \equiv{} & (1-\alpha)\theta(\varepsilon^n - \varepsilon_{10}^*) + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) \\ & + (1-\alpha)\theta \frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\ & - \iota \Bigg\{ \frac{\bar{\varepsilon}}{\alpha_s} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) \\ & + (1-\alpha)\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \Bigg\}. \end{aligned}$$

As $\alpha \to 0$, the function $\tilde{T}(\cdot; \alpha)$ converges uniformly to

$$\begin{aligned} \tilde{T}(\varepsilon_{10}^*; 0) \equiv{} & \theta(\varepsilon^n - \varepsilon_{10}^*) + (1-\theta)\int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) + \theta\frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\ & - \iota \Bigg\{ \frac{\bar{\varepsilon}}{\alpha_s} + (1-\theta)\int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) \\ & + \theta\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \Bigg\}. \end{aligned}$$

Then (72) is equivalent to $\tilde{T}(\varepsilon_{10}^{*};0)=0$, while (69), (70), and (71) are obtained from (209), (207), and (208), respectively, by taking the limit as $\alpha \to 0$.

(*ii*) For $\iota \in (0, \hat{\varsigma}(0)]$ and α small enough, part (*ii*) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon^{*}, \chi, \varphi, \mathcal{Z})$ that satisfies $\chi = \frac{\lambda}{1-\lambda}\frac{1-G(\varepsilon^{*})}{G(\varepsilon^{*})}$,

$$\varphi = \bar{\varepsilon} + \alpha_{s}\left[\int_{\varepsilon_{L}}^{\varepsilon^{*}}(\varepsilon^{*}-\varepsilon)\,dG(\varepsilon) + (1-\alpha)\,\theta\frac{\lambda}{1-\lambda}\int_{\varepsilon^{*}}^{\varepsilon_{H}}(\varepsilon-\varepsilon^{*})\,dG(\varepsilon)\right] \tag{210}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^{*}) + (1-\alpha)\frac{1}{1-\lambda}\left[G(\varepsilon^{*})-\lambda\right]}{\left[1-G(\varepsilon^{*})\right]\left[\alpha+(1-\alpha)\frac{1}{1-\lambda}\right]}\varphi, \tag{211}$$

and $\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\equiv\varepsilon^{*}$, where $\varepsilon^{*}\in[\varepsilon^{n},\varepsilon_{H})$ (with $\varepsilon^{*}=\varepsilon^{n}$ only if $\iota=\hat{\varsigma}(0)$) is the unique solution to $\tilde{\mathcal{T}}(\varepsilon^{*};\alpha)=0$, where for any $\varepsilon^{*}\in[\varepsilon_{L},\varepsilon_{H}]$, $\tilde{\mathcal{T}}(\cdot;\alpha)$ is a real-valued function defined by

$$\begin{aligned}\tilde{\mathcal{T}}(\varepsilon^{*};\alpha) \equiv & \left\{\alpha+(1-\alpha)\left[1+(1-\iota)\,\theta\frac{\lambda}{1-\lambda}\right]\right\}\int_{\varepsilon^{*}}^{\varepsilon_{H}}(\varepsilon-\varepsilon^{*})\,dG(\varepsilon)\\ & -\iota\left[\frac{\bar{\varepsilon}}{\alpha_{s}}+\int_{\varepsilon_{L}}^{\varepsilon^{*}}(\varepsilon^{*}-\varepsilon)\,dG(\varepsilon)\right].\end{aligned}$$

As $\alpha \to 0$, the function $\tilde{\mathcal{T}}(\cdot;\alpha)$ converges uniformly to

$$\tilde{\mathcal{T}}(\varepsilon^{*};0) \equiv \left[1+(1-\iota)\,\theta\frac{\lambda}{1-\lambda}\right]\int_{\varepsilon^{*}}^{\varepsilon_{H}}(\varepsilon-\varepsilon^{*})\,dG(\varepsilon)-\iota\left[\frac{\bar{\varepsilon}}{\alpha_{s}}+\int_{\varepsilon_{L}}^{\varepsilon^{*}}(\varepsilon^{*}-\varepsilon)\,dG(\varepsilon)\right].$$

Then (76) is equivalent to $\tilde{\mathcal{T}}(\varepsilon^{*};0)=0$, while while (73), (74), and (75) are obtained from (211), (207), and (210), respectively, by taking the limit as $\alpha \to 0$. ■

Proof of Proposition 9. The limits for ε_{10}^{*}, φ, $\mathcal{Z}/\varphi$, and $\mathcal{V}$ in part (*i*) are obtained from (50), (51), (52), and (60). The limits for ε^{*}, φ, $\mathcal{Z}/\varphi$, and $\mathcal{V}$ in part (*ii*) are obtained from (53), (54), (55), and (60). ■

A.8 Capital accumulation

Definition 6 *A sequential nonmonetary equilibrium for the economy with investment is an allocation* $\left\{X_{t}, A_{t+1}^{s}\right\}_{t=0}^{\infty}$ *and a sequence of prices,* $\{\varepsilon_{t}^{n}, \phi_{t}^{s}, \bar{\phi}_{t}^{s}\}_{t=0}^{\infty}$*, that satisfy:* $\bar{\phi}_{t}^{s}=\varepsilon_{t}^{n}y_{t}+\phi_{t}^{s}$*, the law of motion for the capital stock,*

$$A_{t+1}^{s}=\eta\left(A_{t}^{s}+X_{t}\right),$$

the market-clearing condition for bonds

$$1 = \left[1 - G\left(\varepsilon_t^n\right)\right] \frac{\bar{\phi}_t^s}{\bar{\phi}_t^s - \lambda \phi_t^s},$$

and the individual optimality conditions

$$X_t = X_t\left(\phi_t^s\right)$$

and

$$\phi_t^s = \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + \alpha_{11}\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon_{t+1}^n}\left(\varepsilon_{t+1}^n - \varepsilon\right)y_{t+1}dG(\varepsilon)$$
$$+ \frac{\lambda\phi_{t+1}^s}{\bar{\phi}_{t+1}^s - \lambda\phi_{t+1}^s}\int_{\varepsilon_{t+1}^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^n\right)y_{t+1}dG(\varepsilon)\Bigg]\Bigg\}.$$

Notice that the structure of the equilibrium conditions in Definition 6 is recursive, i.e., one can solve for $\{\varepsilon_t^n, \phi_t^s\}_{t=0}^{\infty}$ independently of $\left\{X_t, A_{t+1}^s\right\}_{t=0}^{\infty}$, and then given $\{\phi_t^s\}_{t=0}^{\infty}$, one gets $\{X_t\}_{t=0}^{\infty} = \{X_t\left(\phi_t^s\right)\}_{t=0}^{\infty}$, and given $\{X_t\}_{t=0}^{\infty}$, $\left\{A_{t+1}^s\right\}_{t=0}^{\infty}$ follows from the law of motion for the capital stock. Moreover, notice the equations that characterize $\{\varepsilon_t^n, \phi_t^s\}_{t=0}^{\infty}$ in this economy with endogenous capital accumulation are identical to the conditions that characterize $\{\varepsilon_t^n, \phi_t^s\}_{t=0}^{\infty}$ in the baseline economy that assumes $A_t^s = A^s$ for all t.

Definition 7 *A sequential monetary equilibrium for the economy with investment is an allocation $\left\{X_t, A_{t+1}^s\right\}_{t=0}^{\infty}$ and a sequence of prices, $\{\varepsilon_{10t}^*, \varepsilon_{11t}^*, p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$, that satisfy: $\varepsilon_{11t}^* = \left(p_t\frac{1}{q_t} - \phi_t^s\right)\frac{1}{y_t}$, $\varepsilon_{10t}^* = \left(p_t\phi_t^m - \phi_t^s\right)\frac{1}{y_t}$, $\chi_{11} \equiv \chi\left(1,1\right) \in [0,1]$, the law of motion for capital,*

$$A_{t+1}^s = \eta\left(A_t^s + X_t\right),$$

the market clearing conditions for equity and bonds,

$$0 = \alpha_{10}\left[1 - G\left(\varepsilon_{10t}^*\right)\right]\frac{A_t^m + p_t A_t^s}{p_t} + \alpha_{11}\left[1 - G\left(\varepsilon_{11t}^*\right)\right]\frac{A_t^m + p_t A_t^s}{p_t - \lambda q_t \phi_t^s} - \left(\alpha_{10} + \alpha_{11}\right)A_t^s$$
$$0 = \left[1 - \mathbb{I}_{\{q_t\phi_t^m = 1\}}\left(1 - \chi_{11}\right)\right]G\left(\varepsilon_{11t}^*\right) - \frac{\lambda q_t\phi_t^s}{p_t - \lambda q_t\phi_t^s}\left[1 - G\left(\varepsilon_{11t}^*\right)\right],$$

and the individual optimality conditions,

$$X_t = X_t\left(\phi_t^s\right),$$

and

$$\phi_t^m = \beta \mathbb{E}_t \Bigg\{ \phi_{t+1}^m + (\alpha_{01} + \alpha_{11}) \theta \left(\frac{1}{q_{t+1}} - \phi_{t+1}^m \right)$$
$$+ [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_{10t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{10t+1}^* \right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon)$$
$$+ \alpha_{11} \theta \frac{1}{p_{t+1} - \lambda q_{t+1} \phi_{t+1}^s} \int_{\varepsilon_{11t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11t+1}^* \right) y_{t+1} dG(\varepsilon) \Bigg\}$$
$$\phi_t^s = \beta \eta \mathbb{E}_t \Bigg\{ \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10t+1}^*} \left(\varepsilon_{10t+1}^* - \varepsilon \right) y_{t+1} dG(\varepsilon)$$
$$+ \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon_{11t+1}^*} \left(\varepsilon_{11t+1}^* - \varepsilon \right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1} \phi_{t+1}^s}{p_{t+1} - \lambda q_{t+1} \phi_{t+1}^s} \int_{\varepsilon_{11t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11t+1}^* \right) y_{t+1} dG(\varepsilon) \right] \Bigg\}.$$

Notice that the structure of the equilibrium conditions in Definition 7 is recursive, i.e., one can solve for prices and marginal valuations independently of $\left\{X_t, A_{t+1}^s\right\}_{t=0}^{\infty}$, and then given $\{\phi_t^s\}_{t=0}^{\infty}$, one gets $\{X_t\}_{t=0}^{\infty} = \{X_t(\phi_t^s)\}_{t=0}^{\infty}$, and given $\{X_t\}_{t=0}^{\infty}$, $\left\{A_{t+1}^s\right\}_{t=0}^{\infty}$ follows from the law of motion for the capital stock.

Example 1 *Suppose*

$$f_t(n) = \varpi_t n^{\sigma} \tag{212}$$

for $\sigma \in (0,1)$. Then the optimal amount of general goods that the investor devotes to the production of capital goods is

$$g_t(\phi_t^s) = (\sigma \varpi_t \phi_t^s)^{\frac{1}{1-\sigma}} \tag{213}$$

and the quantity of new capital created by an individual investor is

$$x_t(\phi_t^s) = \sigma^{\frac{\sigma}{1-\sigma}} \varpi_t^{\frac{1}{1-\sigma}} (\phi_t^s)^{\frac{\sigma}{1-\sigma}}. \tag{214}$$

Assume

$$\varpi_t = (\sigma y_t)^{-\sigma}. \tag{215}$$

(i) Consider the baseline discrete-time formulation. Given $\phi_t^s = \phi^s y_t$, (213) and (214) with (215) imply

$$g_t(\phi_t^s) = \sigma (\phi^s)^{\frac{1}{1-\sigma}} y_t$$
$$x_t(\phi_t^s) = (\phi^s)^{\frac{\sigma}{1-\sigma}}. \tag{216}$$

(ii) Consider the generalized discrete-time economy with period length Δ. Given the asset price is $\Phi_t^s(\Delta) = \Phi^s(\Delta) y_t \Delta$, (213) and (214) with (215) imply

$$g_t\left(\Phi_t^s(\Delta)\right) = \sigma\left[\Phi^s(\Delta)\Delta\right]^{\frac{1}{1-\sigma}} y_t$$
$$x_t\left(\Phi_t^s(\Delta)\right) = \left[\Phi^s(\Delta)\Delta\right]^{\frac{\sigma}{1-\sigma}}$$

and therefore, since $\lim_{\Delta\to 0}\Phi^s(\Delta)\Delta = \phi^s$,

$$\lim_{\Delta\to 0} g_t\left(\Phi_t^s(\Delta)\right) = \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} y_t \tag{217}$$
$$\lim_{\Delta\to 0} x_t\left(\Phi_t^s(\Delta)\right) = \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}. \tag{218}$$

Thus, in the continuous-time approximation, (217) and (218) are the effort rate devoted to investment, and the investment rate, respectively.

Proof of Proposition 10. Notice the equations that characterize prices and marginal valuations in Definitions 6 and 7 are identical to the conditions that characterize prices and marginal valuations in the baseline economy that assumes $A_t^s = A^s$ for all t (and where only investors carry assets overnight). Hence, the conditions that characterize prices and marginal valuations in the recursive equilibrium, and in the recursive equilibrium with $\Delta \to 0$, are also the same in the economy with endogenous capital accumulation as in the economy that assumes $A_t^s = A^s$ for all t. Given the production function (212) with (215), the aggregate investment rate is immediate from (218). ■

A.9 Unsecured credit

In this section we develop the model with unsecured credit outlined in Section 5.2.

The bargaining solutions for investors of type 10 are as before. The bargaining solutions for investors of type 11 are summarized in the following two results.

Lemma 16 *Consider the economy with no money. If the investor is able to contact both an equity and a bond broker, the post-trade portfolio is*

$$\overline{a}_{11t}^s\left(a_t^s, \varepsilon\right) = \chi\left(\varepsilon_t^n, \varepsilon\right)\left(a_t^s + \frac{\bar{B}_t}{\bar{\phi}_t^s}\right) \tag{219}$$
$$\overline{a}_{11t}^b\left(a_t^s, \varepsilon\right) = \bar{\phi}_t^s\left[a_t^s - \chi\left(\varepsilon_t^n, \varepsilon\right)\left(a_t^s + \frac{\bar{B}_t}{\bar{\phi}_t^s}\right)\right] \tag{220}$$

and the intermediation fee for the bond broker is

$$k_{11t}\left(a_t^s,\varepsilon\right)=\left(1-\theta\right)\left(\varepsilon-\varepsilon_t^n\right)y_t\left[\chi\left(\varepsilon_t^n,\varepsilon\right)\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right)-a_t^s\right]. \tag{221}$$

Proof. In a nonmonetary economy, (5) implies $\left[\overline{a}_{11t}^s\left(a_t^s,\varepsilon\right),\overline{a}_{11t}^b\left(a_t^s,\varepsilon\right),k_{11t}\left(a_t^s,\varepsilon\right)\right]$ is the solution to

$$\max_{(\overline{a}_t^s,k_t)\in\mathbb{R}_+^2,\overline{a}_t^b\in\mathbb{R}}\left[\left(\varepsilon y_t+\phi_t^s\right)\left(\overline{a}_t^s-a_t^s\right)+\overline{a}_t^b-k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{\phi}_t^s\overline{a}_t^s+\overline{a}_t^b=\bar{\phi}_t^s a_t^s \tag{222}$$

$$-\bar{B}_t\leq\overline{a}_t^b. \tag{223}$$

Notice that the first-order condition with respect to k_t implies (136), so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{\overline{a}_t^s\in\mathbb{R}_+,\overline{a}_t^b\in\mathbb{R}}\left[\left(\varepsilon y_t+\phi_t^s\right)\left(\overline{a}_t^s-a_t^s\right)+\overline{a}_t^b\right]\text{ s.t. (222), and (223).}$$

Since (222) implies $\overline{a}_t^b=\bar{\phi}_t^s(a_t^s-\overline{a}_t^s)$,

$$\overline{a}_{11t}^s\left(a_t^s,\varepsilon\right)=\arg\max_{\overline{a}_t^s}\left(\varepsilon-\varepsilon_t^n\right)\overline{a}_t^s\text{ s.t. }0\leq\overline{a}_t^s\text{ and }\overline{a}_t^s\leq a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}.$$

The solution is given by (219). Given $\overline{a}_{11t}^s\left(a_t^s,\varepsilon\right)$, $\overline{a}_{11t}^b\left(a_t^s,\varepsilon\right)=\bar{\phi}_t^s\left[a_t^s-\overline{a}_{11t}^s\left(a_t^s,\varepsilon\right)\right]$ as in (220), and $k_{11t}\left(a_t^s,\varepsilon\right)$ is given by (136), or equivalently, (221). ■

Lemma 17 *Consider the economy with money, and let* $\bar{\varepsilon}_{11t}^*=\max\left(\varepsilon_{10t}^*,\varepsilon_{11t}^*\right)$*, where*

$$\varepsilon_{11t}^*\equiv\frac{p_t\frac{1}{q_t}-\phi_t^s}{y_t} \tag{224}$$

and ε_{10t}^* *is as defined in (19). Consider an investor who enters the OTC round of period t with portfolio* $\boldsymbol{a}_t$ *and valuation* ε *in an economy with money. If the investor is able to contact both* *an equity broker and a bond broker, the post-trade portfolio is*

$$\overline{a}^m_{11t}(\boldsymbol{a}_t,\varepsilon) = \Big\{\mathbb{I}_{\{1<q_t\phi^m_t\}}\left[\mathbb{I}_{\{\varepsilon<\bar{\varepsilon}^*_{11t}\}} + \mathbb{I}_{\{\varepsilon=\bar{\varepsilon}^*_{11t}\}}\chi(\bar{\varepsilon}^*_{11t},\varepsilon)\right]$$
$$+ \mathbb{I}_{\{q_t\phi^m_t=1\}}\mathbb{I}_{\{\varepsilon<\bar{\varepsilon}^*_{11t}\}}\chi(q_t\phi^m_t,1)\Big\}\left(a^m_t + p_t a^s_t + q_t\bar{B}_t\right)$$
$$+ \mathbb{I}_{\{q_t\phi^m_t=1\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}^*_{11t}\}}\hat{a}^m_t \tag{225}$$

$$\overline{a}^s_{11t}(\boldsymbol{a}_t,\varepsilon) = \Big\{\mathbb{I}_{\{\bar{\varepsilon}^*_{11t}<\varepsilon\}} + \left[1-\mathbb{I}_{\{q_t\phi^m_t=1\}}\right]\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}^*_{11t}\}}\chi(\bar{\varepsilon}^*_{11t},\varepsilon)\Big\}\left[a^s_t + \frac{1}{p_t}\left(a^m_t + q_t\bar{B}_t\right)\right]$$
$$+ \mathbb{I}_{\{q_t\phi^m_t=1\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}^*_{11t}\}}\hat{a}^s_t \tag{226}$$

$$\overline{a}^b_{11t}(\boldsymbol{a}_t,\varepsilon) = -\frac{1}{q_t}\left\{\left[\overline{a}^m_{11t}(\boldsymbol{a}_t,\varepsilon) - a^m_t\right] + p_t\left[\overline{a}^s_{11t}(\boldsymbol{a}_t,\varepsilon) - a^s_t\right]\right\}, \tag{227}$$

where

$$(\hat{a}^m_t, \hat{a}^s_t) \in \left\{\mathbb{R}^2_+ : \hat{a}^m_t + p_t\hat{a}^s_t \le a^m_t + p_t a^s_t + q_t\bar{B}_t\right\},$$

and the intermediation fee is

$$k_{11t}(\boldsymbol{a}_t,\varepsilon) = (1-\theta)\Big\{(\varepsilon y_t + \phi^s_t)\left[\overline{a}^s_{11t}(\boldsymbol{a}_t,\varepsilon) - \overline{a}^s_{10t}(\boldsymbol{a}_t,\varepsilon)\right]$$
$$+ \phi^m_t\left[\overline{a}^m_{11t}(\boldsymbol{a}_t,\varepsilon) - \overline{a}^m_{10t}(\boldsymbol{a}_t,\varepsilon)\right] + \overline{a}^b_{11t}(\boldsymbol{a}_t,\varepsilon)\Big\}. \tag{228}$$

Proof. With (128), (5) can be written as

$$\max_{(\overline{a}^m_t,\overline{a}^s_t,k_t)\in\mathbb{R}^3_+,\overline{a}^b_t\in\mathbb{R}} \Big\{(\varepsilon y_t + \phi^s_t)\left[\overline{a}^s_t - \overline{a}^s_{10t}(\boldsymbol{a}_t,\varepsilon)\right] + \phi^m_t\left[\overline{a}^m_t - \overline{a}^m_{10t}(\boldsymbol{a}_t,\varepsilon)\right] + \overline{a}^b_t - k_t\Big\}^\theta k_t^{1-\theta}$$

$$\text{s.t. } \overline{a}^m_t + p_t\overline{a}^s_t + q_t\overline{a}^b_t = a^m_t + p_t a^s_t \tag{229}$$

$$-\bar{B}_t \le \overline{a}^b_t. \tag{230}$$

Notice that the first-order condition with respect to k_t implies (31) so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{(\overline{a}^m_t,\overline{a}^s_t)\in\mathbb{R}^2_+,\overline{a}^b_t\in\mathbb{R}} \Big\{(\varepsilon y_t + \phi^s_t)\left[\overline{a}^s_t - \overline{a}^s_{10t}(\boldsymbol{a}_t,\varepsilon)\right] + \phi^m_t\left[\overline{a}^m_t - \overline{a}^m_{10t}(\boldsymbol{a}_t,\varepsilon)\right] + \overline{a}^b_t\Big\}$$

s.t. (229), and (230).

Once the solution $\overline{a}^m_{11t}(\boldsymbol{a}_t,\varepsilon)$, $\overline{a}^s_{11t}(\boldsymbol{a}_t,\varepsilon)$, and $\overline{a}^b_{11t}(\boldsymbol{a}_t,\varepsilon)$ to this problem has been found, $k_{11t}(\boldsymbol{a}_t,\varepsilon)$ is given by (31). If we use (229) to substitute for $\overline{a}^b_t$, the auxiliary problem is equivalent to

$$\max_{(\overline{a}^m_t,\overline{a}^s_t)\in\mathbb{R}^2_+} \left[\left(\varepsilon y_t + \phi^s_t - \frac{1}{q_t}p_t\right)\overline{a}^s_t + \left(\phi^m_t - \frac{1}{q_t}\right)\overline{a}^m_t\right] \tag{231}$$

$$\text{s.t.} \quad -q_t\bar{B}_t \leq a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right). \tag{232}$$

The Lagrangian corresponding to the auxiliary problem (231) is

$$\mathcal{L} = \left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t\right)\overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t}\right)\overline{a}_t^m$$
$$+ \xi^b\left[a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right) + q_t\bar{B}_t\right] + \xi^m\overline{a}_t^m + \xi^s\overline{a}_t^s,$$

where ξ^b, ξ^m, and ξ^s are the multipliers on the constraints (232), $0 \leq \overline{a}_t^m$, and $0 \leq \overline{a}_t^s$, respectively. The first-order conditions are

$$\varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t + \xi^s - p_t\xi^b = 0$$
$$\phi_t^m - \frac{1}{q_t} + \xi^m - \xi^b = 0.$$

There are eight possible binding patterns for the multipliers $\left(\xi^b, \xi^m, \xi^s\right)$. Case 1. Assume $0 < \xi^m$, $0 < \xi^s$, $0 < \xi^b$. Then $\overline{a}_t^m = \overline{a}_t^s = 0$ and $a_t^m + p_t a_t^s + q_t\bar{B}_t = 0$. Since $0 \leq \bar{B}_t$, this kind of solution has $\overline{a}_t^b = 0$ and is only possible if $a^s = a^m = \bar{B}_t = 0$. Case 2. Assume $0 < \xi^m$, $0 < \xi^s$, $\xi^b = 0$. Then $\overline{a}_t^m = \overline{a}_t^s = 0$, $q_t\overline{a}_t^b = a_t^m + p_t a_t^s$, $\xi^s = \left[\left(\frac{p_t}{q_t} - \phi_t^s\right)\frac{1}{y_t} - \varepsilon\right]y_t$, and $\xi^m = \frac{1}{q_t} - \phi_t^m$. This kind of solution is only possible if $q_t\phi_t^m < 1$ and $\varepsilon y_t < \frac{1}{q_t}p_t - \phi_t^s$. Case 3. Assume $0 < \xi^m$, $\xi^s = 0$, $0 < \xi^b$. Then $\overline{a}_t^m = 0$, $\overline{a}_t^s = a_t^s + \frac{1}{p_t}\left(q_t\bar{B}_t + a_t^m\right)$, $\overline{a}_t^b = -\bar{B}_t$, $p_t\xi^b = \varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t$, and $p_t\xi^m = \varepsilon y_t + \phi_t^s - p_t\phi_t^m$. This kind of solution is only possible if $\max\left(q_t\phi_t^m, 1\right)\frac{1}{q_t}p_t - \phi_t^s < \varepsilon y_t$. Case 4. Assume $\xi^m = 0$, $0 < \xi^s$, $0 < \xi^b$. Then $\overline{a}_t^m = a_t^m + p_t a_t^s + q_t\bar{B}_t$, $\overline{a}_t^s = 0$, $\overline{a}_t^b = -\bar{B}_t$, $\xi^s = p_t\phi_t^m - \phi_t^s - \varepsilon y_t$, and $p_t\xi^b = \left(q_t\phi_t^m - 1\right)\frac{1}{q_t}p_t$. This kind of solution is only possible if $1 < q_t\phi_t^m$ and $\varepsilon y_t < p_t\phi_t^m - \phi_t^s$. Case 5. Assume $0 < \xi^m$, $\xi^s = 0$, $\xi^b = 0$. Then $\overline{a}_t^m = 0$, $\xi^m = \frac{1}{q_t} - \phi_t^m$, and $\left(\overline{a}_t^s, \overline{a}_t^b\right)$ is any pair that satisfies $\left(\overline{a}_t^s, \overline{a}_t^b\right) \in [0,\infty) \times [-\bar{B}_t, \infty)$ and $q_t\overline{a}_t^b + p_t\overline{a}_t^s = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m < 1$ and $\varepsilon y_t = \frac{1}{q_t}p_t - \phi_t^s$. Case 6. Assume $\xi^m = 0$, $\xi^s = 0$, $0 < \xi^b$. Then $p_t\xi^b = \left(q_t\phi_t^m - 1\right)\frac{1}{q_t}p_t = \varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t$, $\left(\overline{a}_t^m, \overline{a}_t^s\right)$ is any pair that satisfies $\left(\overline{a}_t^m, \overline{a}_t^s\right) \in [0,\infty) \times [0,\infty)$ and $a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right) + q_t\bar{B}_t = 0$, and $\overline{a}_t^b = -\bar{B}_t$. This kind of solution is only possible if $1 < q_t\phi_t^m$ and $\varepsilon y_t = p_t\phi_t^m - \phi_t^s$. Case 7. Assume $\xi^m = 0$, $0 < \xi^s$, $\xi^b = 0$. Then $\overline{a}_t^s = 0$, $\xi^s = \frac{1}{q_t}p_t - \phi_t^s - \varepsilon y_t$, and $\left(\overline{a}_t^m, \overline{a}_t^b\right)$ is any pair that satisfies $\left(\overline{a}_t^m, \overline{a}_t^b\right) \in [0,\infty) \times [-\bar{B}_t, \infty)$ and $\overline{a}_t^m + q_t\overline{a}_t^b = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m = 1$ and $\varepsilon y_t < \frac{1}{q_t}p_t - \phi_t^s$. Case 8. Assume $\xi^m = 0$, $\xi^s = 0$, $\xi^b = 0$. Then $\left(\overline{a}_t^m, \overline{a}_t^s, \overline{a}_t^b\right) \in [0,\infty) \times [0,\infty) \times [-\bar{B}_t, \infty)$ is any triple that satisfies $\overline{a}_t^m + p_t\overline{a}_t^s + q_t\overline{a}_t^b = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m = 1$ and $\varepsilon y_t = \frac{1}{q_t}p_t - \phi_t^s$. By collecting the solutions along with the inequality restrictions implied by the eight cases, we obtain (225)-(228). ■

Next, we derive the market-clearing conditions for equity and bonds in the OTC round, in a nonmonetary economy (Lemma 18), and in a monetary economy (Lemma 19).

Lemma 18 *In a nonmonetary equilibrium, the market-clearing condition for equity,* $\bar{A}^s_{Et} + \bar{A}^s_{10t} + \bar{A}^s_{11t} = (\alpha_{10} + \alpha_{11}) A^s$ *(or bonds,* $\bar{A}^b_{Bt} + \bar{A}^b_{11t} = 0$*) in the OTC round is:*

$$1 = [1 - G(\varepsilon^n_t)] \left(1 + \frac{N_I \bar{B}_t}{\bar{\phi}^s_t A^s}\right) \tag{233}$$

where

$$\Lambda_t \equiv N_I \bar{B}_t. \tag{234}$$

Proof. The aggregate post-trade holdings of equity for agents who trade in the equity market in the OTC round of period t are

$$\begin{aligned}
\bar{A}^s_{Et} &= A^s_{Et} = 0 \\
\bar{A}^s_{11t} &= \alpha_{11} N_I \int \bar{a}^s_{11t}(a_t, \varepsilon) dH_{It}(a_t, \varepsilon) = \alpha_{11} [1 - G(\varepsilon^n_t)] \left(A^s + \frac{N_I \bar{B}_t}{\bar{\phi}^s_t}\right) \\
\bar{A}^s_{10t} &= \alpha_{10} N_I \int \bar{a}^s_{10t}(a_t, \varepsilon) dH_{It}(a_t, \varepsilon) = \alpha_{10} A^s
\end{aligned}$$

and the aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\begin{aligned}
\bar{A}^b_{Bt} &= N_B \int \bar{a}^b_{Bt}(a_t)\, dF_{Bt}(a_t) = 0 \\
\bar{A}^b_{11t} &= \alpha_{11} N_I \int \bar{a}^b_{11t}(a_t, \varepsilon) dH_{It}(a_t, \varepsilon) = \alpha_{11} \bar{\phi}^s_t \left[A^s - [1 - G(\varepsilon^n_t)] \left(A^s + \frac{N_I \bar{B}_t}{\bar{\phi}^s_t}\right)\right].
\end{aligned}$$

■

Lemma 19 *In a monetary equilibrium, the market-clearing conditions for equity,* $\bar{A}^s_{Et} + \bar{A}^s_{10t} + \bar{A}^s_{11t} = (\alpha_{10} + \alpha_{11}) A^s_{It}$*, and bonds,* $\bar{A}^b_{Bt} + \bar{A}^b_{11t} = 0$*, in the OTC round are, respectively:*

$$\begin{aligned}
0 &= \alpha_{10} [1 - G(\varepsilon^*_{10t})] \frac{A^m_t + p_t A^s}{p_t} + \alpha_{11} [1 - G(\bar{\varepsilon}^*_{11t})] \frac{A^m_t + p_t A^s + q_t N_I \bar{B}_t}{p_t} \\
&\quad - (\alpha_{10} + \alpha_{11}) A^s \\
0 &= \left\{G(\bar{\varepsilon}^*_{11t}) \left[\mathbb{I}_{\{1 < q_t \phi^m_t\}} + \mathbb{I}_{\{q_t \phi^m_t = 1\}} \chi(q_t \phi^m_t, 1)\right] + 1 - G(\bar{\varepsilon}^*_{11t})\right\} \frac{A^m_t + p_t A^s + q_t N_I \bar{B}_t}{p_t} \\
&\quad - \left(A^s + \frac{A^m_t}{p_t}\right).
\end{aligned}$$

Proof. The aggregate post-trade holdings of equity for agents who trade in the equity market in the OTC round of period t are

$$\bar{A}^s_{Et} = N_E \int \bar{a}^s_{Et}(\boldsymbol{a}_t)\, dF_{Et}(\boldsymbol{a}_t) = \chi(\varepsilon^*_{10t}, 0)\, \frac{A^m_{Et} + p_t A^s_{Et}}{p_t} = 0$$

$$\bar{A}^s_{11t} = \alpha_{11} N_I \int \overline{a}^s_{11t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = \alpha_{11}\left[1 - G(\bar{\varepsilon}^*_{11t})\right]\left[A^s + \frac{1}{p_t}\left(A^m_t + q_t N_I \bar{B}_t\right)\right]$$

$$\bar{A}^s_{10t} = \alpha_{10} N_I \int \overline{a}^s_{10t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = \alpha_{10}\left[1 - G(\varepsilon^*_{10t})\right] \frac{A^m_t + p_t A^s}{p_t}$$

and the the aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\bar{A}^b_{Bt} = N_B \int \bar{a}^b_{Bt}(\boldsymbol{a}_t)\, dF_{Bt}(\boldsymbol{a}_t) = \left[1 - \chi(1, q_t\phi^m_t)\right] \frac{1}{q_t} A^m_{Bt} - \chi(1, q_t\phi^m_t)\, \lambda \phi^s_t A^s_{Bt} = 0$$

$$\bar{A}^b_{11t} = \alpha_{11} N_I \int \overline{a}^b_{11t}(\boldsymbol{a}_t, \varepsilon) dH_{It}(\boldsymbol{a}_t, \varepsilon) = -\frac{p_t}{q_t}\alpha_{11}\left\{\left\{G(\bar{\varepsilon}^*_{11t})\left[\mathbb{I}_{\{1<q_t\phi^m_t\}} + \mathbb{I}_{\{q_t\phi^m_t=1\}}\chi(q_t\phi^m_t, 1)\right]\right.\right.$$

$$\left.\left. + 1 - G(\bar{\varepsilon}^*_{11t})\right\} \frac{A^m_t + p_t A^s + q_t N_I \bar{B}_t}{p_t} - \left(A^s + \frac{A^m_t}{p_t}\right)\right\}.$$

■

The following result states that the credit market would be inactive if the net nominal interest rate on bonds, $i^m_t \equiv \frac{1}{q_t\phi^m_t} - 1$, were negative.

Lemma 20 *Consider a monetary equilibrium. If the bond market is active in period t, then $q_t\phi^m_t \leq 1$.*

Proof. In an equilibrium with $1 < q_t\phi^m_t$, the bond-market clearing condition in Lemma 19 becomes

$$0 = \left[A^s + \frac{1}{p_t}\left(A^m_t + q_t N_I \bar{B}_t\right)\right] - \left(A^s + \frac{A^m_t}{p_t}\right).$$

This condition can only hold if $\bar{B}_t = 0$, i.e., if the bond market is inactive at all dates. The condition $1 < q_t\phi^m_t$ implies bond demand is nil, so the bond market can only clear with no trade. ■

In what follows, we focus on monetary equilibria with an active credit market, i.e., equilibria with $q_t\phi^m_t \leq 1$. Notice this implies $\bar{\varepsilon}^*_{11t} = \varepsilon^*_{11t}$ for all t in any monetary equilibrium.

Next, we derive an investor's value function in a nonmonetary economy (Lemma 21), and in a monetary economy (Lemma 22).

Lemma 21 *Consider an economy without money. The value function of an investor who enters the OTC round of period t with equity holding a_t^s and valuation ε is*

$$V_t^I(a_t^s,\varepsilon) = \left[\varepsilon y_t + \phi_t^s + \alpha_{11}\theta\mathbb{I}_{\{\varepsilon<\varepsilon_t^n\}}(\varepsilon_t^n-\varepsilon)y_t\right]a_t^s + \tilde{W}_t^I(\varepsilon), \tag{235}$$

where

$$\tilde{W}_t^I(\varepsilon) \equiv \bar{W}_t^I + \alpha_{11}\theta\mathbb{I}_{\{\varepsilon_t^n<\varepsilon\}}(\varepsilon-\varepsilon_t^n)y_t\frac{\bar{B}_t}{\bar{\phi}_t^s}$$
$$\bar{W}_t^I \equiv \max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left[-\phi_t^s\tilde{a}_{t+1}^s + \beta\mathbb{E}_t\int V_{t+1}^I\left[\eta\tilde{a}_{t+1}^s+(1-\eta)A^s,\varepsilon\right]dG(\varepsilon)\right].$$

Proof. With (153), and Lemma 16, (11) reduces to

$$V_t^I(a_t^s,\varepsilon) = \bar{W}_t^I + (\varepsilon y_t+\phi_t^s)a_t^s$$
$$+\alpha_{11}\theta(\varepsilon-\varepsilon_t^n)y_t\left[\mathbb{I}_{\{\varepsilon_t^n<\varepsilon\}}\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right)-a_t^s\right],$$

which can be written as (235). ■

Lemma 22 *Consider an economy with money. The value function of an investor who enters the OTC round of period t with portfolio $\boldsymbol{a}_t$ and valuation ε is*

$$V_t^I(\boldsymbol{a}_t,\varepsilon) = v_{It}^m(\varepsilon)a_t^m + v_{It}^s(\varepsilon)a_t^s + \tilde{W}_t^I(\varepsilon), \tag{236}$$

where

$$v_{It}^m(\varepsilon) \equiv \phi_t^m + [\alpha_{10}+\alpha_{11}(1-\theta)]\mathbb{I}_{\left\{\varepsilon_{10t}^*<\varepsilon\right\}}(\varepsilon-\varepsilon_{10t}^*)y_t\frac{1}{p_t}$$
$$+\alpha_{11}\theta(\varepsilon-\bar{\varepsilon}_{11t}^*)y_t\mathbb{I}_{\left\{\bar{\varepsilon}_{11t}^*<\varepsilon\right\}}\frac{1}{p_t}$$
$$+\alpha_{11}\theta\left(\frac{1}{q_t}-\phi_t^m\right)\left\{\mathbb{I}_{\{q_t\phi_t^m<1\}}+\mathbb{I}_{\{1<q_t\phi_t^m\}}\mathbb{I}_{\left\{\varepsilon=\bar{\varepsilon}_{11t}^*\right\}}\left[1-2\chi(\bar{\varepsilon}_{11t}^*,\varepsilon)\right]\right\}$$
$$v_{It}^s(\varepsilon) \equiv \varepsilon y_t + \phi_t^s + [\alpha_{10}+\alpha_{11}(1-\theta)](\varepsilon_{10t}^*-\varepsilon)y_t\mathbb{I}_{\left\{\varepsilon<\varepsilon_{10t}^*\right\}}$$
$$+\alpha_{11}\theta(\bar{\varepsilon}_{11t}^*-\varepsilon)y_t\mathbb{I}_{\left\{\varepsilon<\bar{\varepsilon}_{11t}^*\right\}}$$
$$+\alpha_{11}\theta\left(\frac{1}{q_t}-\phi_t^m\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\mathbb{I}_{\left\{\varepsilon=\bar{\varepsilon}_{11t}^*\right\}}\left[1-2\chi(\bar{\varepsilon}_{11t}^*,\varepsilon)\right]p_t$$
$$\tilde{W}_t^I(\varepsilon) \equiv \bar{W}_t^I + \alpha_{11}\theta\left\{(\varepsilon-\bar{\varepsilon}_{11t}^*)y_t\mathbb{I}_{\left\{\bar{\varepsilon}_{11t}^*<\varepsilon\right\}}\frac{1}{p_t}\right.$$
$$\left.+\left(\phi_t^m-\frac{1}{q_t}\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\left\{1+\mathbb{I}_{\left\{\varepsilon=\bar{\varepsilon}_{11t}^*\right\}}\left[2\chi(\bar{\varepsilon}_{11t}^*,\varepsilon)-1\right]\right\}\right\}q_t\bar{B}_t.$$

Proof. With (128), the value function (11) becomes (146), which after substituting $k_{11t}(\boldsymbol{a}_t, \varepsilon)$, $k_{01t}(\boldsymbol{a}_t, \varepsilon)$, $\bar{a}^b_{01t}(\boldsymbol{a}_t, \varepsilon)$, and $\bar{a}^b_{11t}(\boldsymbol{a}_t, \varepsilon)$ with (228), (138), $\bar{a}^b_{01t}(\boldsymbol{a}_t, \varepsilon) = -\frac{1}{q_t}[\bar{a}^m_{01t}(\boldsymbol{a}_t, \varepsilon) - a^m_t]$, and (227), respectively, becomes

$$
\begin{aligned}
V^I_t(\boldsymbol{a}_t, \varepsilon) = \bar{W}^I_t &+ (\varepsilon y_t + \phi^s_t) a^s_t + \phi^m_t a^m_t \\
&+ [\alpha_{10} + \alpha_{11}(1-\theta)] \{(\varepsilon y_t + \phi^s_t) [\overline{a}^s_{10t}(\boldsymbol{a}_t, \varepsilon) - a^s_t] + \phi^m_t [\bar{a}^m_{10t}(\boldsymbol{a}_t, \varepsilon) - a^m_t]\} \\
&+ \alpha_{11}\theta \left\{ \left(\varepsilon y_t + \phi^s_t - \frac{1}{q_t} p_t \right) [\bar{a}^s_{11t}(\boldsymbol{a}_t, \varepsilon) - a^s_t] + \left(\phi^m_t - \frac{1}{q_t} \right) [\bar{a}^m_{11t}(\boldsymbol{a}_t, \varepsilon) - a^m_t] \right\}.
\end{aligned}
$$

Then replace the post-trade allocations $\overline{a}^s_{10t}(\boldsymbol{a}_t, \varepsilon)$ and $\bar{a}^m_{10t}(\boldsymbol{a}_t, \varepsilon)$ (using Lemma 2), and $\bar{a}^s_{11t}(\boldsymbol{a}_t, \varepsilon)$, and $\bar{a}^m_{11t}(\boldsymbol{a}_t, \varepsilon)$ (using Lemma 17), and rearrange terms to arrive at (236). ■

Next, we derive the Euler equations that characterize the investor's optimal portfolio choices in the second subperiod, in a nonmonetary economy (Lemma 23) and in a nonmonetary economy (Lemma 24).

Lemma 23 *Consider an economy with no money. Let $\tilde{a}^s_{It+1}$ denote equity holding chosen by an investor in the second subperiod of period t. Then $\tilde{a}^s_{It+1}$ is optimal if and only if it satisfies*

$$\phi^s_t \geq \beta\eta\mathbb{E}_t \left[\bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n_{t+1}} \left(\varepsilon^n_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \right] \text{ with "= " if } \tilde{a}^s_{It+1} > 0.$$

Proof. With (235), the portfolio problem of an investor in the second subperiod (i.e., the maximization on the right side of (8)) can be written as

$$\max_{\tilde{a}^s_{t+1} \in \mathbb{R}_+} \left\{ -\phi^s_t + \beta\eta\mathbb{E}_t \left[\bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n_{t+1}} \left(\varepsilon^n_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \right] \right\} \tilde{a}^s_{t+1}.$$

■

Lemma 24 *Consider an economy with money. Let $\left(\tilde{a}^m_{It+1}, \tilde{a}^s_{It+1}\right)$ denote the portfolio choice of an investor in the second subperiod of period t. The portfolio $\left(\tilde{a}^m_{It+1}, \tilde{a}^s_{It+1}\right)$ is optimal if and only if it satisfies*

$$\left(\phi^m_t - \beta\mathbb{E}_t \bar{v}^m_{It+1}\right) \tilde{a}^m_{It+1} = 0 \leq \phi^m_t - \beta\mathbb{E}_t \bar{v}^m_{It+1} \tag{237}$$

$$\left(\phi^s_t - \beta\eta\mathbb{E}_t \bar{v}^s_{It+1}\right) \tilde{a}^s_{It+1} = 0 \leq \phi^s_t - \beta\eta\mathbb{E}_t \bar{v}^s_{It+1}, \tag{238}$$

where

$$\bar{v}_{It+1}^{m} \equiv \phi_{t+1}^{m} + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_{10t+1}^{*}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{10t+1}^{*}\right) y_{t+1} dG(\varepsilon) \frac{1}{p_{t+1}}$$
$$+ \alpha_{11}\theta \frac{1}{p_{t+1}} \int_{\bar{\varepsilon}_{11t+1}^{*}}^{\varepsilon_H} \left(\varepsilon - \bar{\varepsilon}_{11t+1}^{*}\right) y_{t+1} dG(\varepsilon)$$
$$+ \alpha_{11}\theta \left(\frac{1}{q_{t+1}} - \phi_{t+1}^{m}\right) \mathbb{I}_{\left\{q_{t+1}\phi_{t+1}^{m}<1\right\}}$$

and

$$\bar{v}_{It+1}^{s} \equiv \bar{\varepsilon} y_{t+1} + \phi_{t+1}^{s} + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10t+1}^{*}} \left(\varepsilon_{10t+1}^{*} - \varepsilon\right) y_{t+1} dG(\varepsilon)$$
$$+ \alpha_{11}\theta \int_{\varepsilon_L}^{\bar{\varepsilon}_{11t+1}^{*}} \left(\bar{\varepsilon}_{11t+1}^{*} - \varepsilon\right) y_{t+1} dG(\varepsilon).$$

Proof. With (236), the portfolio problem of an equity broker in the second subperiod (i.e., the maximization on the right side of (8)) can be written as

$$\max_{\tilde{\boldsymbol{a}}_{t+1}\in\mathbb{R}_{+}^{2}} \left[-\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t \left(\bar{v}_{It+1}^m \tilde{a}_{t+1}^m + \eta \bar{v}_{It+1}^s \tilde{a}_{t+1}^s\right)\right],$$

where $\bar{v}_{It+1}^{k} \equiv \int v_{It+1}^{k}(\varepsilon)\, dG(\varepsilon)$ for $k \in \{m, s\}$. ■

Next, we define sequential nonmonetary equilibrium and monetary equilibrium (with an active credit market).

Definition 8 *A (sequential) nonmonetary equilibrium is a sequence* $\{\varepsilon_t^n, \phi_t^s, \bar{\phi}_t^s\}_{t=0}^{\infty}$*, that satisfies*

$$0 = [1 - G(\varepsilon_t^n)] \left(A^s + \frac{N_I \bar{B}_t}{\varepsilon_t^n y_t + \phi_t^s}\right) - A^s$$
$$\phi_t^s = \beta\eta \mathbb{E}_t \left[\bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon_{t+1}^n} \left(\varepsilon_{t+1}^n - \varepsilon\right) y_{t+1} dG(\varepsilon)\right]$$
$$\bar{\phi}_t^s = \varepsilon_t^n y_t + \phi_t^s.$$

The first condition in Definition 8 is the bond-market clearing condition (233), the second is the investor's Euler equation from Lemma 23, and the last is the definition of ε_t^n (13).

Definition 9 *A (sequential) monetary equilibrium is a sequence* $\{\varepsilon^*_{10t}, \varepsilon^*_{11t}, p_t, q_t, \phi^m_t, \phi^s_t\}_{t=0}^{\infty}$*, that satisfy* $\varepsilon^*_{11t} = (p_t \frac{1}{q_t} - \phi^s_t)\frac{1}{y_t}$*,* $\varepsilon^*_{10t} = (p_t\phi^m_t - \phi^s_t)\frac{1}{y_t}$*,* $\chi^B_L \equiv \chi(1,1) \in [0,1]$*, the market clearing conditions,*

$$
\begin{aligned}
0 &= \alpha_{10}\left[1 - G\left(\varepsilon^*_{10t}\right)\right]\frac{A^m_t + p_t A^s}{p_t} + \alpha_{11}\left[1 - G\left(\varepsilon^*_{11t}\right)\right]\frac{A^m_t + p_t A^s + q_t N_I \bar{B}_t}{p_t} \\
&\quad - \left(\alpha_{10} + \alpha_{11}\right) A^s \\
0 &= \left\{G\left(\varepsilon^*_{11t}\right)\left[\mathbb{I}_{\{1<q_t\phi^m_t\}} + \mathbb{I}_{\{q_t\phi^m_t=1\}}\chi^B_L\right] + \left[1 - G\left(\varepsilon^*_{11t}\right)\right]\right\}\frac{A^m_t + p_t A^s + q_t N_I \bar{B}_t}{p_t} \\
&\quad - \left(A^s + \frac{A^m_t}{p_t}\right),
\end{aligned}
$$

the Euler equations,

$$
\begin{aligned}
\phi^m_t &= \beta\mathbb{E}_t\Bigg\{\phi^m_{t+1} + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right]\int_{\varepsilon^*_{10t+1}}^{\varepsilon_H}\left(\varepsilon - \varepsilon^*_{10t+1}\right) y_{t+1} dG(\varepsilon)\frac{1}{p_{t+1}} \\
&\quad + \alpha_{11}\theta\frac{1}{p_{t+1}}\int_{\varepsilon^*_{11t+1}}^{\varepsilon_H}\left(\varepsilon - \varepsilon^*_{11t+1}\right) y_{t+1} dG(\varepsilon) \\
&\quad + \alpha_{11}\theta\left(\frac{1}{q_{t+1}} - \phi^m_{t+1}\right)\mathbb{I}_{\{q_{t+1}\phi^m_{t+1}<1\}}\Bigg\} \\
\phi^s_t &= \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right]\int_{\varepsilon_L}^{\varepsilon^*_{10t+1}}\left(\varepsilon^*_{10t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
&\quad + \alpha_{11}\theta\int_{\varepsilon_L}^{\varepsilon^*_{11t+1}}\left(\varepsilon^*_{11t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon)\Bigg\}.
\end{aligned}
$$

Next, we define RNE and RME (with an active credit market). To this end, hereafter we assume $\bar{B}_t$ is as defined in (94). As before, a RNE is a nonmonetary equilibrium in which real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e., $\phi^s_t = \phi^s y_t$ and $\bar{\phi}^s_t = \bar{\phi}^s y_t$ for some $\phi^s, \bar{\phi}^s \in \mathbb{R}_+$. Hence in a RNE, $\varepsilon^n_t = \left(\bar{\phi}^s_t - \phi^s_t\right)\frac{1}{y_t} = \bar{\phi}^s - \phi^s \equiv \varepsilon^n$. Similarly, a RME is a monetary equilibrium in which: (i) real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e., $\phi^s_t = \phi^s y_t$, $p_t\phi^m_t \equiv \bar{\phi}^s_{10t} = \bar{\phi}^s_{10} y_t$, and $p_t/q_t \equiv \bar{\phi}^s_{11t} = \bar{\phi}^s_{11} y_t$ for some $\phi^s, \bar{\phi}^s_{10}, \bar{\phi}^s_{11} \in \mathbb{R}_+$; and (ii) real money balances are a constant proportion of output, i.e., $\phi^m_t A^m_t = Z A^s y_t$ for some $Z \in \mathbb{R}_{++}$. Hence in a RME, $\varepsilon^*_{10t} = \left(p_t\phi^m_t - \phi^s_t\right)\frac{1}{y_t} = \bar{\phi}^s_{10} - \phi^s \equiv \varepsilon^*_{10}$, $\varepsilon^*_{11t} = \left(p_t/q_t - \phi^s_t\right)\frac{1}{y_t} = \bar{\phi}^s_{11} - \phi^s \equiv \varepsilon^*_{11}$, $p_t = \frac{\left(\varepsilon^*_{10} + \phi^s\right)A^m_t}{ZA^s}$, $\phi^m_t = \frac{ZA^s y_t}{A^m_t}$, and q_t is given by (36).

Definition 10 *A recursive nonmonetary equilibrium of the economy with borrowing limit (94), is a triple* $(\varepsilon^n, \phi^s, \bar{\phi}^s)$*, that satisfies* $\bar{\phi}^s = \varepsilon^n + \phi^s$*,*

$$0 = [1 - G(\varepsilon^n)](1 + \Lambda) - 1$$
$$\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta}\phi^s = \bar{\varepsilon} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon).$$

Definition 11 *A recursive monetary equilibrium of the economy with borrowing limit (94), is a vector* $(\varepsilon_{10}^*, \varepsilon_{11}^*, \phi^s, Z, \chi_L^B)$ *that satisfies* $\chi_L^B \in [0, 1]$*, and*

$$\begin{aligned}
0 &= \alpha_{10}\left[1 - G(\varepsilon_{10}^*)\right]\left(1 + \frac{Z}{\varepsilon_{10}^* + \phi^s}\right) + \alpha_{11}\left[1 - G(\varepsilon_{11}^*)\right]\left(1 + \Lambda + \frac{Z}{\varepsilon_{10}^* + \phi^s}\right) \\
&\quad - (\alpha_{10} + \alpha_{11}) \\
0 &= \left\{G(\varepsilon_{11}^*)\,\mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}}\chi_L^B + \left[1 - G(\varepsilon_{11}^*)\right]\right\}\left(1 + \Lambda + \frac{Z}{\varepsilon_{10}^* + \phi^s}\right) \\
&\quad - \left(1 + \frac{Z}{\varepsilon_{10}^* + \phi^s}\right) \\
i^p &= \left[\alpha_{10} + \alpha_{11}(1 - \theta)\right]\frac{1}{\varepsilon_{10}^* + \phi^s}\int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) \\
&\quad + \alpha_{11}\theta\frac{1}{\varepsilon_{10}^* + \phi^s}\int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) + \alpha_{11}\theta\frac{\varepsilon_{11}^* - \varepsilon_{10}^*}{\varepsilon_{10}^* + \phi^s}\mathbb{I}_{\{\varepsilon_{10}^* < \varepsilon_{11}^*\}} \\
\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta}\phi^s &= \bar{\varepsilon} + \left[\alpha_{10} + \alpha_{11}(1 - \theta)\right]\int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon).
\end{aligned}$$

In a nonmonetary equilibrium, $p_t/q_t = \bar{\phi}_t^s \equiv \varepsilon_t^n y_t + \phi_t^s$, and therefore the borrowing limit (94) becomes

$$\bar{B}_t \equiv \Lambda\frac{(\varepsilon_t^n y_t + \phi_t^n)\, A^s}{N_I}. \tag{239}$$

In a monetary equilibrium, $p_t/q_t = \varepsilon_{11t}^* y_t + \phi_t^s$, and therefore the borrowing limit (94) becomes

$$\bar{B}_t \equiv \Lambda\frac{(\varepsilon_{11t}^* y_t + \phi_t^s)\, A^s}{N_I}. \tag{240}$$

In the discrete-time economy with period length equal to Δ, (239) generalizes to

$$\bar{B}_t(\Delta) = \Lambda\frac{\left[\varepsilon_t^n y_t \Delta + \Phi_t^n(\Delta)\right] A^s}{N_I} \tag{241}$$

and (240) generalizes to

$$\bar{B}_t \equiv \Lambda\frac{\left[\varepsilon_{11t}^* y_t \Delta + \Phi_t^s(\Delta)\right] A^s}{N_I}. \tag{242}$$

In a RNE, $\varepsilon_t^n = \varepsilon^n$ and $\Phi_t^n(\Delta) = \Phi^n(\Delta)\, y_t \Delta$, so (241) specializes to

$$\bar{B}_t(\Delta) = \Lambda \frac{[\varepsilon^n + \Phi^n(\Delta)]\, A^s}{N_I} y_t \Delta.$$

In a RME, $\varepsilon_{11t}^* = \varepsilon_{11}^*$ and $\Phi_t^s(\Delta) = \Phi^s(\Delta)\, y_t \Delta$, so (242) specializes to

$$\bar{B}_t(\Delta) = \Lambda \frac{[\varepsilon_{11}^* + \Phi^s(\Delta)]\, A^s}{N_I} y_t \Delta.$$

Next, we report the equilibrium conditions for the continuous-time limiting economy as $\Delta \to 0$.

Lemma 25 *Consider the limiting economy (as $\Delta \to 0$) with borrowing limit (94). A recursive nonmonetary equilibrium is a pair (ε^n, φ) that satisfies*

$$G(\varepsilon^n) = \frac{\Lambda}{1+\Lambda}$$
$$\varphi = \bar{\varepsilon} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon).$$

Proof. The first equilibrium condition is immediate from the first condition in Definition 10. The second condition is obtained by recognizing that, in a discrete-time economy with period length Δ, the second condition in Definition 10 is

$$\frac{r + \delta - g + g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)} \Phi^s(\Delta)\, \Delta = \bar{\varepsilon} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)$$

and letting $\Delta \to 0$. ■

Lemma 26 *Consider the limiting economy (as $\Delta \to 0$) with borrowing limit (94). A recursive monetary equilibrium is a vector $(\varepsilon_{10}^*, \varepsilon_{11}^*, \varphi, \mathcal{Z}, \chi_L^B)$ that satisfies $\chi_L^B \in [0,1]$, and*

$$0 = \alpha_{10}\left[1 - G(\varepsilon_{10}^*)\right]\left(1 + \frac{\mathcal{Z}}{\varphi}\right) + \alpha_{11}\left[1 - G(\varepsilon_{11}^*)\right]\left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - (\alpha_{10} + \alpha_{11})$$

$$0 = \left\{G(\varepsilon_{11}^*)\, \mathbb{I}_{\{\varepsilon_{10}^* = \varepsilon_{11}^*\}} \chi_L^B + \left[1 - G(\varepsilon_{11}^*)\right]\right\}\left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - \left(1 + \frac{\mathcal{Z}}{\varphi}\right)$$

$$\iota\varphi = \left[\alpha_{10} + \alpha_{11}(1-\theta)\right]\int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon)$$
$$+ \alpha_{11}\theta\left[(\varepsilon_{11}^* - \varepsilon_{10}^*)\, \mathbb{I}_{\{\varepsilon_{10}^* < \varepsilon_{11}^*\}} + \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon)\right]$$

$$\varphi = \bar{\varepsilon} + \left[\alpha_{10} + \alpha_{11}(1-\theta)\right]\int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon).$$

Proof. In a discrete-time economy with period length Δ, the equilibrium conditions in Definition 11 generalize to

$$\begin{aligned}0 &= \alpha_{10}\left[1-G\left(\varepsilon_{10}^{*}\right)\right]\left(1+\frac{\mathrm{Z}\left(\Delta\right)\Delta}{\varepsilon_{10}^{*}\Delta+\Phi^{s}\left(\Delta\right)\Delta}\right)\\ &+\alpha_{11}\left[1-G\left(\varepsilon_{11}^{*}\right)\right]\left[1+\Lambda+\frac{\mathrm{Z}\left(\Delta\right)\Delta}{\varepsilon_{10}^{*}\Delta+\Phi^{s}\left(\Delta\right)\Delta}\right]\\ &-\left(\alpha_{10}+\alpha_{11}\right)\end{aligned}$$

$$\begin{aligned}0 = &\left\{G\left(\varepsilon_{11}^{*}\right)\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\chi_{L}^{B}+\left[1-G\left(\varepsilon_{11}^{*}\right)\right]\right\}\left[1+\Lambda+\frac{\mathrm{Z}\left(\Delta\right)\Delta}{\varepsilon_{10}^{*}\Delta+\Phi^{s}\left(\Delta\right)\Delta}\right]\\ &-\left(1+\frac{\mathrm{Z}\left(\Delta\right)\Delta}{\varepsilon_{10}^{*}\Delta+\Phi^{s}\left(\Delta\right)\Delta}\right)\end{aligned}$$

$$\begin{aligned}\frac{i^{p}}{\Delta}\Phi^{s}\left(\Delta\right)\Delta = \frac{\Phi^{s}\left(\Delta\right)\Delta}{\varepsilon_{10}^{*}\Delta+\Phi^{s}\left(\Delta\right)\Delta}&\Bigg\{\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{10}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{10}^{*}\right)dG(\varepsilon)\\ &+\alpha_{11}\theta\left[\left(\varepsilon_{11}^{*}-\varepsilon_{10}^{*}\right)\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_{11}^{*}\right\}}+\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG(\varepsilon)\right]\Bigg\}\end{aligned}$$

$$\begin{aligned}\frac{r+\delta-g+g\delta\Delta}{\left(1+g\Delta\right)\left(1-\delta\Delta\right)}\Phi^{s}\left(\Delta\right)\Delta = \bar{\varepsilon}&+\left[\alpha_{10}+\alpha_{11}\left(1-\theta\right)\right]\int_{\varepsilon_{L}}^{\varepsilon_{10}^{*}}\left(\varepsilon_{10}^{*}-\varepsilon\right)dG\left(\varepsilon\right)\\ &+\alpha_{11}\theta\int_{\varepsilon_{L}}^{\varepsilon_{11}^{*}}\left(\varepsilon_{11}^{*}-\varepsilon\right)dG\left(\varepsilon\right).\end{aligned}$$

Take the limit as $\Delta \to 0$ to obtain the conditions in the statement of the lemma. ■

Proof of Proposition 11. As $\alpha \to 0$, the equilibrium conditions in Lemma 26 become

$$0 = \left[1-G\left(\varepsilon_{11}^{*}\right)\right]\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right)-1 \tag{243}$$

$$0 = \left\{G\left(\varepsilon_{11}^{*}\right)\mathbb{I}_{\left\{\varepsilon_{10}^{*}=\varepsilon_{11}^{*}\right\}}\chi_{L}^{B}+\left[1-G\left(\varepsilon_{11}^{*}\right)\right]\right\}\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right)-\left(1+\frac{\mathcal{Z}}{\varphi}\right) \tag{244}$$

$$\begin{aligned}\iota\varphi = \alpha_{s}\left(1-\theta\right)&\int_{\varepsilon_{10}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{10}^{*}\right)dG(\varepsilon)+\alpha_{s}\theta\int_{\varepsilon_{11}^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon_{11}^{*}\right)dG(\varepsilon)\\ &+\alpha_{s}\theta\left(\varepsilon_{11}^{*}-\varepsilon_{10}^{*}\right)\mathbb{I}_{\left\{\varepsilon_{10}^{*}<\varepsilon_{11}^{*}\right\}}\end{aligned} \tag{245}$$

$$\varphi = \bar{\varepsilon}+\alpha_{s}\left(1-\theta\right)\int_{\varepsilon_{L}}^{\varepsilon_{10}^{*}}\left(\varepsilon_{10}^{*}-\varepsilon\right)dG\left(\varepsilon\right)+\alpha_{s}\theta\int_{\varepsilon_{L}}^{\varepsilon_{11}^{*}}\left(\varepsilon_{11}^{*}-\varepsilon\right)dG\left(\varepsilon\right) \tag{246}$$

where $\chi_L^B \in [0,1]$. These are four equations in four unknowns. The unknowns are $(\varepsilon_{10}^*, \varepsilon_{11}^*, \phi^s, Z)$ if $\varepsilon_{10}^* < \varepsilon_{11}^*$, or $(\varepsilon^*, \chi_L^B, \phi^s, Z)$ if $\varepsilon_{10}^* = \varepsilon_{11}^*$. We consider each case in turn.

(i) Suppose $\varepsilon_{10}^* < \varepsilon_{11}^*$. In this case, (243) and (244) imply $\frac{\mathcal{Z}}{\varphi} = 0$ and $\varepsilon_{11}^* = \varepsilon^n$. Combined, conditions (245) and (246) imply a single equation in the unknown ε_{10}^* that can be written as $T(\varepsilon_{10}^*) = 0$, where

$$\begin{aligned} T(x) \equiv\ & \alpha_s \left[\theta(\varepsilon^n - x) + (1-\theta)\int_x^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon) + \theta \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \\ & - \iota \left[\bar{\varepsilon} + \alpha_s (1-\theta) \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon) + \alpha_s \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)\right]. \end{aligned}$$

Differentiate T and evaluate the derivative at $x = \varepsilon_{10}^*$ to obtain

$$T'(\varepsilon_{10}^*) = -\alpha_s \left\{\theta + (1-\theta)\left[1 - G(\varepsilon_{10}^*) + \iota G(\varepsilon_{10}^*)\right]\right\} < 0.$$

Hence, if there is a ε_{10}^* that satisfies $T(\varepsilon_{10}^*) = 0$, it is unique. Notice that

$$\begin{aligned} T(\varepsilon_L) = & \alpha_s \left[\theta(\varepsilon^n - \varepsilon_L) + (1-\theta)(\bar{\varepsilon} - \varepsilon_L) + \theta \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \\ & - \iota \left[\bar{\varepsilon} + \alpha_{11}\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)\right], \end{aligned}$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\varsigma}_0$. Also,

$$T(\varepsilon^n) = \alpha_s \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \alpha_s \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)\right],$$

so $T(\varepsilon^n) < 0$ if and only if $\hat{\varsigma}_0 < \iota$. Thus if $\hat{\varsigma}_0 < \iota < \bar{\varsigma}_0$, there exists a unique ε_{10}^* that satisfies $T(\varepsilon_{10}^*) = 0$, and $\varepsilon_{10}^* \in (\varepsilon_L, \varepsilon^n)$. Given ε_{10}^* and ε_{11}^*, φ is given by (243).

(ii) Suppose $\varepsilon_{10}^* = \varepsilon_{11}^* \equiv \varepsilon^*$. In this case, (243)-(246) become

$$0 = [1 - G(\varepsilon^*)]\left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - 1 \tag{247}$$

$$0 = \left\{G(\varepsilon^*)\chi_L^B + [1 - G(\varepsilon^*)]\right\}\left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - \left(1 + \frac{\mathcal{Z}}{\varphi}\right) \tag{248}$$

$$\iota\varphi = \alpha_s \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \tag{249}$$

$$\varphi = \bar{\varepsilon} + \alpha_s \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon). \tag{250}$$

Combined, conditions (249) and (250) imply a single equation in the unknown ε^* that can be written as $\mathcal{T}(\varepsilon^*) = 0$, where

$$\mathcal{T}(x) \equiv \alpha_s \int_x^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \alpha_s \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon)\right].$$

Differentiate $\mathcal{T}$ and evaluate the derivative at $x = \varepsilon^*$ to obtain

$$\mathcal{T}'(\varepsilon^*) = -\alpha_s \left[1 - G(\varepsilon^*) + \iota G(\varepsilon^*)\right] < 0.$$

Hence, if there is a ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, it is unique. Notice that

$$\mathcal{T}(\varepsilon^n) = \alpha_s \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \alpha_s \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)\right],$$

so $0 \leq \mathcal{T}(\varepsilon^n)$ if and only if $\iota \leq \hat{\varsigma}_0$. Also,

$$\mathcal{T}(\varepsilon_H) = -\iota \left[\bar{\varepsilon} + \alpha_s (\varepsilon_H - \bar{\varepsilon})\right] \leq 0, \text{ with "} = \text{" only if } \iota = 0.$$

Thus, if $0 < \iota \leq \hat{\varsigma}_0$, there exists a unique ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, and $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\varsigma}_0$). Given ε^*, φ is given by (250). Given ε^* and φ, (247) implies

$$\mathcal{Z} = \frac{G(\varepsilon^*) - [1 - G(\varepsilon^*)]\,\Lambda}{1 - G(\varepsilon^*)}\varphi.$$

Finally, given, ε^*, φ, and $\mathcal{Z}$, (248) implies

$$\chi_L^B = 1 - \frac{1 - G(\varepsilon^*)}{G(\varepsilon^*)}\Lambda.$$

This concludes the proof. ■

A.10 Efficiency

Proof of Proposition 12. The choice variable $\bar{a}_t^E$ does not appear in the planner's objective function, so $\bar{a}_t^E = 0$ at an optimum. Also, (98) must bind for every t at an optimum, so the planner's problem is equivalent to

$$\max_{\{\tilde{a}_{t+1}^I, \bar{a}_t^I\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(\alpha_{10} + \alpha_{11}) \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}_t^I (d\varepsilon) + (1 - \alpha_{10} - \alpha_{11}) \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t a_t^I dG(\varepsilon)\right] N_I$$

$$\text{s.t. (96), (99), and } \int_{\varepsilon_L}^{\varepsilon_H} \bar{a}_t^I (d\varepsilon) \leq a_t^I.$$

Then clearly,

$$W^*(y_0) \leq [\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})] A^s \left(\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t \right). \tag{251}$$

The allocation consisting of $\tilde{a}^I_t = A^s/N_I$ and the Dirac measure $\bar{a}^I_t(E) = \frac{A^s}{N_I} \mathbb{I}_{\{\varepsilon_H \in E\}}$ defined in the statement of the proposition achieve the value on the right side of (251) and therefore solve the planner's problem. Notice that $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t = \frac{\bar{\beta}}{1-\bar{\beta}} y_0$, so

$$W^*(y_0) = \frac{\bar{\beta}}{1-\bar{\beta}} [\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})] A^s y_0.$$

Hence in the discrete-time economy with period of length Δ, welfare is

$$\mathcal{W}^*(y_0) = \frac{1 + g\Delta}{(r-g)\Delta} [\bar{\varepsilon} + (\alpha_{10} + \alpha_{11})(\varepsilon_H - \bar{\varepsilon})] A^s y_0 \Delta.$$

Rearrange this expression and take the limit as $\Delta \to 0$ to arrive at (100). ■

Proof of Proposition 13. The choice variable $\bar{a}^E_t$ does not appear in the planner's objective function, so $\bar{a}^E_t = 0$ at an optimum. Also, since (117) must bind for every t at an optimum, the planner's problem is equivalent to

$$\max_{\{\tilde{a}^I_{t+1}, \bar{a}^I_t, h^I_{2t}, X_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ (\alpha_{10} + \alpha_{11}) \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}^I_t(d\varepsilon)$$
$$+ (1 - \alpha_{10} - \alpha_{11}) \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t a^I_t dG(\varepsilon) - h^I_{2t} \Bigg\} N_I$$
$$\text{s.t. (114), (115), (116), and } \int_{\varepsilon_L}^{\varepsilon_H} \bar{a}^I_t(d\varepsilon) \leq a^I_t.$$

Clearly,

$$W^*(A^s_0, y_0) \leq \max_{\{h^I_{2t}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\psi A^s_t y_t - h^I_{2t} N_I \right) \text{ s.t. } A^s_{t+1} = \eta \left[A^s_t + f_t\left(h^I_{2t}\right) N_I \right], \tag{252}$$

where ψ is defined in (101). Once $\left\{h^I_{2t}\right\}_{t=0}^{\infty}$ has been found, we can use (116) to get $X_t = f_t\left(h^I_{2t}\right) N_I$, and (114) at equality to get $\tilde{a}^I_{t+1} = \frac{A^s_t + X_t}{N_I}$. Let $\bar{W}^*(A_0, y_0)$ denote the value of the right side of (252); it satisfies

$$\bar{W}^*(A^s_t, y_t) = \max_{0 \leq h} \left[\psi A^s_t y_t - h N_I + \beta \mathbb{E}_t \bar{W}^*\left(A^s_{t+1}, y_{t+1}\right) \right] \tag{253}$$
$$\text{s.t. } A^s_{t+1} = \eta \left[A^s_t + f_t(h) N_I \right].$$

It is easy to show the optimal value function that satisfies (253) is $\bar{W}^*(A_t^s, y_t) = (BA_t^s + C)\, y_t$, where

$$B = \frac{\psi}{1-\bar{\beta}\eta}$$
$$C = \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\psi\right)^{\frac{1}{1-\sigma}} N_I.$$

The decision rule implied by (253) is

$$h(y_t) = \sigma\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\psi\right)^{\frac{1}{1-\sigma}} y_t \tag{254}$$

and the implied aggregate investment is

$$f_t[h(y_t)]\, N_I = \left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\psi\right)^{\frac{\sigma}{1-\sigma}} N_I. \tag{255}$$

Hence,

$$\bar{W}^*(A_t^s, y_t) = \left(\frac{\psi}{1-\bar{\beta}\eta}A_t^s + \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\psi\right)^{\frac{1}{1-\sigma}} N_I\right) y_t. \tag{256}$$

The OTC-market allocation consisting of the Dirac measure $\bar{a}_t^I(E) = \frac{A_t^s}{N_I}\mathbb{I}_{\{\varepsilon_H \in E\}}$ defined in the statement of the proposition along with the decision rules (254) and (255) achieve the value on the right side of (252) and therefore solve the planner's problem, i.e., $W^*(A_t^s, y_t) = \bar{W}^*(A_t^s, y_t)$.

Next consider the generalization to a time period of length Δ. In this case, (253) becomes

$$\bar{W}^*(A_t^s, y_t) = \max_{0 \le h}\left[\psi A_t^s y_t \Delta - \Delta h N_I + \beta \mathbb{E}_t \bar{W}^*\left(A_{t+\Delta}^s, y_{t+\Delta}\right)\right] \tag{257}$$
$$\text{s.t. } A_{t+\Delta}^s = \eta\left[A_t^s + \Delta f_t(h)\, N_I\right],$$

where y_t, h, and $f_t(h)$ are now the per-unit-time dividend, effort, and output, respectively. It is easy to verify that the optimal value function is still $\bar{W}^*(A_t^s, y_t) = (BA_t^s + C)\, y_t$ (proportional to the *dividend rate*), but with

$$B = \frac{1}{1-\bar{\beta}\eta}\psi\Delta = \frac{1}{1-\frac{(1+g\Delta)(1-\delta\Delta)}{1+r\Delta}}\psi\Delta = \frac{1+r\Delta}{r+\delta-g+\delta g\Delta}\psi$$
$$C = \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\psi\Delta\right)^{\frac{1}{1-\sigma}} N_I\Delta = \frac{1+r\Delta}{r-g}(1-\sigma)\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r+\delta-g+g\delta\Delta}\psi\right]^{\frac{1}{1-\sigma}} N_I.$$

The decision rule for the effort rate is $h(y_t) = \sigma\left(\bar{\beta}\eta B\right)^{\frac{1}{1-\sigma}} y_t$ and the implied aggregate investment rate is $f_t[h(y_t)]N_I = \left(\bar{\beta}\eta B\right)^{\frac{\sigma}{1-\sigma}}$, or explicitly,

$$h(y_t) = \sigma\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r-g+\delta+g\delta\Delta}\psi\right]^{\frac{1}{1-\sigma}} y_t$$

$$f_t[h(y_t)]N_I = \left[\frac{(1+g\Delta)(1-\delta\Delta)}{r-g+\delta+g\delta\Delta}\psi\right]^{\frac{\sigma}{1-\sigma}} N_I.$$

Hence

$$\bar{W}^*(A_t^s, y_t) = \left\{\frac{1+r\Delta}{r+\delta-g+\delta g\Delta}\psi A_t^s + \frac{1+r\Delta}{r-g}(1-\sigma)\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r+\delta-g+g\delta\Delta}\psi\right]^{\frac{1}{1-\sigma}} N_I\right\} y_t.$$

Take the limit as $\Delta \to 0$ and let $\mathcal{W}^*(A_t^s, y_t) = \lim_{\Delta\to 0}\bar{W}^*(A_t^s, y_t)$ to arrive at (119). ■

Proof of Proposition 14. From Proposition 2, we know that $\varepsilon_{10}^* = \varepsilon_{11}^* \equiv \varepsilon^* \to \varepsilon_H$, and $\varphi \to \psi$ as $\iota \to 0$. ■

A.11 Equilibrium welfare

The following result characterizes equilibrium welfare for the economy with exogenous capital.

Lemma 27 *Consider the limiting economy as $\Delta \to 0$ with exogenous capital. Along the path of the recursive equilibrium, we have:*

(i) If the equilibrium is nonmonetary, the welfare function is

$$\mathcal{V}^n(y_t) = \frac{\varphi_1^n}{r-g}A^s y_t \tag{258}$$

with

$$\varphi_1^n \equiv \bar{\varepsilon} + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right].$$

(ii) If the equilibrium is monetary, the welfare function is

$$\mathcal{V}^m(\mathcal{Z}, y_t) = \frac{1}{r-g}\left(u_1^z\frac{\mathcal{Z}}{\varphi} + \bar{\varepsilon} + u_1^s\right)A^s y_t, \tag{259}$$

where

$$u_1^z \equiv \alpha_{10}\int_{\varepsilon_{10}^*}^{\varepsilon_H}(\varepsilon - \varepsilon_{10}^*)\,dG(\varepsilon) + \alpha_{11}\left[\varepsilon_{11}^* - \varepsilon_{10}^* + \frac{1}{1-\lambda}\int_{\varepsilon_{11}^*}^{\varepsilon_H}(\varepsilon - \varepsilon_{11}^*)\,dG(\varepsilon)\right]$$

$$u_1^s \equiv \alpha_{10}\int_{\varepsilon_L}^{\varepsilon_{10}^*}(\varepsilon_{10}^* - \varepsilon)\,dG(\varepsilon) + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon_{11}^*}(\varepsilon_{11}^* - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon_{11}^*}^{\varepsilon_H}(\varepsilon - \varepsilon_{11}^*)\,dG(\varepsilon)\right].$$

Proof. (i) Consider an economy with no money. From (151), the beginning-of-period expected discounted utility of an investor along a recursive equilibrium where he holds a^s equity shares at the beginning of every period is

$$\int V_t^I\left(a^s,\varepsilon\right)dG\left(\varepsilon\right)=\left\{\bar{\varepsilon}+\alpha_{11}\theta\left[\frac{\varepsilon^n+\phi^n}{\varepsilon^n+\left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)-\left(\bar{\varepsilon}-\varepsilon^n\right)\right]\right\}a^s y_t$$
$$+\beta\mathbb{E}_t\int V_{t+1}^I\left(a^s,\varepsilon\right)dG(\varepsilon).$$

Notice we can write $\int V_t^I\left(a^s,\varepsilon\right)dG\left(\varepsilon\right)=\bar{V}^I\left(a^s\right)y_t$, where $\bar{V}^I\left(a^s\right)$ is given by

$$\left(1-\bar{\beta}\right)\bar{V}^I\left(a^s\right)=\left\{\bar{\varepsilon}+\alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda\phi^n}{\varepsilon^n+\left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}a^s. \tag{260}$$

Since there are N_I investors, along a recursive equilibrium path each investor is holding $a^s=A^s/N_I$, and the sum of expected utility across all investors is $N_I\bar{V}^I\left(A^s/N_I\right)y_t=\bar{V}^I\left(A^s\right)y_t$.

The expected discounted utility of a bond broker at the beginning of a period is given by (147). Since in this environment brokers hold no assets overnight, we have $V_t^B\left(a_t^s\right)=V_t^B\left(0\right)\equiv V_t^B$ and $\bar{W}_t^B=\beta\mathbb{E}_tV_{t+1}^B$ for all t, where V_t^B satisfies

$$V_t^B=\alpha_{11}^B\int k_{11t}\left(a_t^s,\varepsilon\right)dH_{It}\left(a_t^s,\varepsilon\right)+\beta\mathbb{E}_tV_{t+1}^B. \tag{261}$$

Since there are N_B bond brokers, the sum of expected utility across all bond brokers is

$$N_BV_t^B=\alpha_{11}^BN_B\int k_{11t}\left(a_t^s,\varepsilon\right)dH_{It}\left(a_t^s,\varepsilon\right)+\beta\mathbb{E}_tN_BV_{t+1}^B$$
$$=\alpha_{11}N_I\int k_{11t}\left(a_t^s,\varepsilon\right)dH_{It}\left(a_t^s,\varepsilon\right)+\beta\mathbb{E}_tN_BV_{t+1}^B.$$

From (17), $N_I\int k_{11t}\left(a_t^s,\varepsilon\right)dH_{It}\left(a_t^s,\varepsilon\right)=\bar{\Xi}\left(A^s\right)y_t$, where

$$\bar{\Xi}\left(A^s\right)\equiv\left(1-\theta\right)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda\phi^n}{\varepsilon^n+\left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]A^s. \tag{262}$$

Hence, we can write $N_BV_t^B=\bar{V}^B\left(A^s\right)y_t$ and therefore (261) implies

$$\left(1-\bar{\beta}\right)\bar{V}^B\left(A^s\right)=\alpha_{11}\bar{\Xi}\left(A^s\right). \tag{263}$$

Along a RNE path, total welfare can be written as $V_t = \sum_{k \in \{B,I\}} \bar{V}^k (A^s) y_t$ (equity brokers earn no fees so their utility is zero and they contribute nothing to welfare). Combine (260) and (263) to obtain

$$V_t = \frac{1}{1-\bar{\beta}} \left\{ \bar{\varepsilon} + \alpha_{11} \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) \, dG(\varepsilon) + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda) \phi^n} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) \, dG(\varepsilon) \right] \right\} A^s y_t.$$

In the discrete-time economy with time-period of length Δ, the expression for V_t generalizes to

$$\begin{aligned} V_t = \frac{1 + r\Delta}{(r-g)\Delta} \Bigg\{ & \bar{\varepsilon} + \alpha_{11} \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) \, dG(\varepsilon) \\ & + \frac{\lambda \Phi^n(\Delta)}{\varepsilon^n + (1-\lambda) \Phi^n(\Delta)} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) \, dG(\varepsilon) \Bigg] \Bigg\} A^s y_t \Delta. \end{aligned}$$

Take the limit as $\Delta \to 0$ and let $\mathcal{V}^n(y_t) \equiv \lim_{\Delta \to 0} V_t$ to arrive at (258).

(ii) Consider a monetary economy. From (145), the beginning-of-period expected welfare of an investor along a recursive equilibrium where he holds portfolio (a_t^m, a^s) at the beginning of every period is

$$\int V_t^I (a_t^m, a^s, \varepsilon) \, dG(\varepsilon) = \bar{v}_{It}^m a_t^m + \bar{v}_{It}^s a^s + \bar{W}_t^I, \tag{264}$$

where $\bar{W}_t^I$ is given by (131), and $\bar{v}_{It}^m$ and $\bar{v}_{It}^s$ are defined in Lemma 5 and can be written as

$$\begin{aligned} \bar{v}_{It}^m &= \bar{v}^z \frac{1}{p_t} y_t \\ \bar{v}_{It}^s &= \bar{v}^s y_t, \end{aligned}$$

where

$$\begin{aligned} \bar{v}^z \equiv\ & \varepsilon_{10}^* + \phi^s + \alpha_{11} \theta (\varepsilon_{11}^* - \varepsilon_{10}^*) \\ & + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*) \, dG(\varepsilon) \\ & + \alpha_{11} \theta \frac{\varepsilon_{11}^* + \phi^s}{\varepsilon_{11}^* + (1-\lambda)\phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*) \, dG(\varepsilon) \end{aligned} \tag{265}$$

$$\begin{aligned} \bar{v}^s \equiv\ & \bar{\varepsilon} + \phi^s + [\alpha_{10} + \alpha_{11}(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon) \, dG(\varepsilon) \\ & + \alpha_{11} \theta \left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon) \, dG(\varepsilon) + \frac{\lambda \phi^s}{\varepsilon_{11}^* + (1-\lambda)\phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*) \, dG(\varepsilon) \right]. \end{aligned} \tag{266}$$

Along the path of a recursive equilibrium an individual investor is holding portfolio $(a^m_{t+1}, a^s) = \left(A^m_{t+1}/N_I, A^s/N_I\right)$ at the end of period t (and at the beginning of period $t+1$). Therefore,

$$\bar{W}^I_t = T_t - \phi^m_t \frac{A^m_{t+1}}{N_I} - \phi^s_t \frac{A^s}{N_I} + \beta \mathbb{E}_t \int V^I_{t+1} \left(A^m_{t+1}/N_I, A^s/N_I, \varepsilon\right) dG(\varepsilon). \tag{267}$$

Substitute (267) into (264), and use the government budget constraint, $N_I T_t = \phi^m_t \left(A^m_{t+1} - A^m_t\right)$, to get the sum of expected utility across all investors

$$\begin{aligned} N_I \int V^I_t \left(A^m_t/N_I, A^s/N_I, \varepsilon\right) dG\left(\varepsilon\right) &= \bar{v}^z \frac{1}{p_t} A^m_t y_t + \bar{v}^s y_t A^s - \phi^m_t A^m_t - \phi^s_t A^s \\ &\quad + \beta \mathbb{E}_t N_I \int V^I_{t+1} \left(A^m_{t+1}/N_I, A^s/N_I, \varepsilon\right) dG(\varepsilon). \end{aligned}$$

Then, since in a recursive equilibrium, $p_t = \frac{\left(\varepsilon^*_{10} + \phi^s\right) A^m_t}{Z A^s}$ and $\phi^m_t A^m_t = Z A^s y_t$, we have

$$\begin{aligned} N_I \int V^I_t \left(A^m_t/N_I, A^s/N_I, \varepsilon\right) dG\left(\varepsilon\right) &= \left(\frac{\bar{v}^z}{\varepsilon^*_{10} + \phi^s} - 1\right) Z A^s y_t + \left(\bar{v}^s - \phi^s\right) A^s y_t \\ &\quad + \beta \mathbb{E}_t N_I \int V^I_{t+1} \left(A^m_{t+1}/N_I, A^s/N_I, \varepsilon\right) dG(\varepsilon). \end{aligned}$$

Hence we can write $N_I \int V^I_t \left(A^m_t/N_I, A^s/N_I, \varepsilon\right) dG\left(\varepsilon\right) = \bar{V}^I\left(Z, A^s\right) y_t$, and therefore

$$\bar{V}^I\left(Z, A^s\right) = \left(\frac{\bar{v}^z - \varepsilon^*_{10} - \phi^s}{\varepsilon^*_{10} + \phi^s} Z + \bar{v}^s - \phi^s\right) A^s + \bar{\beta} \bar{V}^I\left(Z, A^s\right)$$

so

$$\left(1 - \bar{\beta}\right) \bar{V}^I\left(Z, A^s\right) = \left(\frac{u^z}{\varepsilon^*_{10} + \phi^s} Z + \bar{\varepsilon} + u^s\right) A^s, \tag{268}$$

where

$$u^z \equiv \bar{v}^z - \left(\varepsilon^*_{10} + \phi^s\right) \tag{269}$$

$$u^s \equiv \bar{v}^s - \left(\bar{\varepsilon} + \phi^s\right), \tag{270}$$

with $\bar{v}^z$ and $\bar{v}^s$ given by (265) and (266).

The expected welfare of a bond broker at the beginning of a period is given by (143). Since in this environment $\alpha_{01} = \alpha^B_{01} = 0$ and bond brokers hold no assets overnight, we have $V^B_t\left(\boldsymbol{a}_t\right) = V^B_t\left(\boldsymbol{0}\right) \equiv V^B_t$ and $\bar{W}^B_t = \beta \mathbb{E}_t V^B_{t+1}$ for all t, where V^B_t satisfies

$$V^B_t = \alpha^B_{11} \int k_{11t}\left(\boldsymbol{a}_t, \varepsilon\right) dH_{It}\left(\boldsymbol{a}_t, \varepsilon\right) + \beta \mathbb{E}_t V^B_{t+1}. \tag{271}$$

Since there are N_B bond brokers, the sum of expected utility across all bond brokers is

$$\begin{aligned} N_B V_t^B &= \alpha_{11}^B N_B \int k_{11t}(\boldsymbol{a}_t, \varepsilon)\, dH_{It}(\boldsymbol{a}_t, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B \\ &= \alpha_{11} N_I \int k_{11t}(\boldsymbol{a}_t, \varepsilon)\, dH_{It}(\boldsymbol{a}_t, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B. \end{aligned}$$

From (31),

$$\begin{aligned} N_I \int k_{11t}(\boldsymbol{a}_t, \varepsilon)\, dH_{It}(\boldsymbol{a}_t, \varepsilon) = (1-\theta) \Bigg[&\frac{p_t}{p_t - \lambda q_t \phi_t^s} \int_{\varepsilon_{11t}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11t}^*)\, dG(\varepsilon) \\ &+ (\varepsilon_{11t}^* - \varepsilon_{10t}^*) - \int_{\varepsilon_{10t}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10t}^*)\, dG(\varepsilon) \Bigg] \frac{1}{p_t} (A_t^m + p_t A^s)\, y_t. \end{aligned}$$

In a recursive equilibrium, $N_I \int k_{11t}(\boldsymbol{a}_t, \varepsilon)\, dH_{It}(\boldsymbol{a}_t, \varepsilon) = \bar{\Xi}(Z, A^s)\, y_t$, where

$$\begin{aligned} \bar{\Xi}(Z, A^s) = (1-\theta) \Bigg[&\frac{\varepsilon_{11}^* + \phi^s}{\varepsilon_{11}^* + (1-\lambda)\phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \\ &+ (\varepsilon_{11}^* - \varepsilon_{10}^*) - \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) \Bigg] \left(\frac{Z}{\varepsilon_{10}^* + \phi^s} + 1 \right) A^s. \end{aligned} \tag{272}$$

Hence we can write $N_B V_t^B = \bar{V}^B(Z, A^s)\, y_t$ and therefore (271) implies

$$\left(1 - \bar{\beta}\right) \bar{V}^B(Z, A^s) = \alpha_{11} \bar{\Xi}(Z, A^s). \tag{273}$$

Notice that (272) can be used to write (269) and (270) as

$$u^z = (\alpha_{10} + \alpha_{11}) \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) + \alpha_{11} \theta \frac{\bar{\Xi}(Z, A^s)}{(1-\theta) \left(\frac{Z}{\varepsilon_{10}^* + \phi^s} + A^s \right)} \tag{274}$$

$$u^s = (\alpha_{10} + \alpha_{11}) \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11} \theta \frac{\bar{\Xi}(Z, A^s)}{(1-\theta) \left(\frac{Z}{\varepsilon_{10}^* + \phi^s} + A^s \right)}. \tag{275}$$

Along a RNE, total welfare is $V_t = \sum_{k \in \{B, I\}} \bar{V}^k(Z, A^s)\, y_t$ (equity brokers earn no fees so their utility is zero and they contribute nothing to welfare). With (268) and (273), we obtain

$$V_t = \frac{1}{1 - \bar{\beta}} \left[\left(\frac{u^z}{\varepsilon_{10}^* + \phi^s} Z + \bar{\varepsilon} + u^s \right) A^s + \alpha_{11} \bar{\Xi}(Z, A^s) \right] y_t$$

and substituting (272), (274) and (275), we arrive at

$$V_t = \frac{1}{1 - \bar{\beta}} \left(\tilde{u}_1^z \frac{Z}{\varepsilon_{10}^* + \phi^s} + \bar{\varepsilon} + \tilde{u}_1^s \right) A^s y_t$$

with

$$\tilde{u}_1^z \equiv \alpha_{10} \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) + \alpha_{11} \left[\varepsilon_{11}^* - \varepsilon_{10}^* + \frac{\varepsilon_{11}^* + \phi^s}{\varepsilon_{11}^* + (1-\lambda)\,\phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \right]$$

$$\tilde{u}_1^s \equiv \alpha_{10} \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11} \left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \phi^s}{\varepsilon_{11}^* + (1-\lambda)\,\phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \right].$$

For the discrete-time formulation with time-period of length Δ, the expression for V_t generalizes to

$$V_t = \frac{1 + r\Delta}{(r-g)\,\Delta} \left[\tilde{u}_1^z(\Delta) \frac{\mathrm{Z}(\Delta)}{\varepsilon_{10}^* + \bar{\Xi}^s(\Delta)} + \bar{\varepsilon} + \tilde{u}_1^s(\Delta) \right] A^s y_t \Delta$$

with

$$\tilde{u}_1^z(\Delta) \equiv \alpha_{10} \int_{\varepsilon_{10}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{10}^*)\, dG(\varepsilon) + \alpha_{11} \left[\varepsilon_{11}^* - \varepsilon_{10}^* + \frac{\varepsilon_{11}^* + \Phi^s(\Delta)}{\varepsilon_{11}^* + (1-\lambda)\,\Phi^s(\Delta)} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \right]$$

$$\tilde{u}_1^s(\Delta) \equiv \alpha_{10} \int_{\varepsilon_L}^{\varepsilon_{10}^*} (\varepsilon_{10}^* - \varepsilon)\, dG(\varepsilon) + \alpha_{11} \left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} (\varepsilon_{11}^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \Phi^s(\Delta)}{\varepsilon_{11}^* + (1-\lambda)\,\Phi^s(\Delta)} \int_{\varepsilon_{11}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{11}^*)\, dG(\varepsilon) \right].$$

Take the limit as $\Delta \to 0$ and let $\mathcal{V}^m(\mathcal{Z}, y_t) \equiv \lim_{\Delta \to 0} V_t$ to arrive at (259). ■

The following result characterizes equilibrium welfare for the economy with capital accumulation with production technology given by (91).

Lemma 28 *Consider the limiting economy (as $\Delta \to 0$) with capital accumulation. Along the path of the recursive equilibrium:*

(i) If the equilibrium is nonmonetary, the welfare function is

$$\mathcal{V}^n(A_t^s, y_t) = \left[\frac{\varphi_1^n}{\rho} A_t^s + \frac{1}{r-g} \left(\frac{\varphi_1^n}{\varphi^n} - \sigma \right) \left(\frac{\varphi^n}{\rho} \right)^{\frac{1}{1-\sigma}} N_I \right] y_t \tag{276}$$

with φ_1^n as defined in part (i) of Lemma 27.

(ii) If the equilibrium is monetary, the welfare function is

$$\mathcal{V}^m\left(\mathcal{Z}, A_t^s, y_t\right) = \frac{1}{r-g}\left\{\left(\frac{u_1^z}{\rho}\frac{\mathcal{Z}}{\varphi} + \frac{\varphi_1}{\rho}\right)\left[(r-g)\,A_t^s + \left(\frac{\varphi}{\rho}\right)^{\frac{\sigma}{1-\sigma}} N_I\right] - \sigma\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right\} y_t \tag{277}$$

with $\varphi_1 \equiv \bar{\varepsilon} + u_1^s$ and u_1^z and u_1^s as defined in part (ii) of Lemma 27.

Proof. (i) Consider an economy with no money. From (151), the sum of expected discounted utility across all investors at the beginning of period t along a recursive equilibrium where each investor holds A_t^s/N_I equity shares, is

$$N_I \int V_t^I\left(A_t^s/N_I, \varepsilon\right) dG\left(\varepsilon\right) = N_I\left\{\bar{\varepsilon} + \phi^n + \alpha_{11}\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right)\right.\right.$$
$$\left.\left.+\frac{\lambda\phi^n}{\varepsilon^n + (1-\lambda)\,\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right)\right]\right\}\frac{A_t^s}{N_I} y_t + N_I \bar{W}_t^I,$$

where

$$\bar{W}_t^I \equiv \max_{h_{2t}\in\mathbb{R}_+}\left[\phi^n y_t f_t\left(h_{2t}\right) - h_{2t}\right]$$
$$+ \max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left[-\phi^n y_t \tilde{a}_{t+1}^s + \beta\mathbb{E}_t \int V_{t+1}^I\left(\eta\tilde{a}_{t+1}^s, \varepsilon\right) dG(\varepsilon)\right].$$

Along a RNE path with $\phi_t^s = \phi^n y_t$, we have $h_{2t} = g_t\left(\phi_t^n\right) = \sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}} y_t$, $f_t\left(h_{2t}\right) = x_t\left(\phi_t^n\right) = \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}$, $\tilde{a}_{t+1}^s = \left(A_t^s + X_t\right)/N_I$, and $X_t = N_I x_t\left(\phi_t^n\right)$, as described in Section 5.1 (where as in Section 3), so

$$\bar{W}_t^I \equiv -\left[\sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}} + \phi^n\frac{A_t^s}{N_I}\right] y_t + \beta\mathbb{E}_t \int V_{t+1}^I\left[\eta\left(A_t^s/N_I + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon).$$

Also, along a recursive equilibrium where each investor holds A_t^s/N_I equity shares at the beginning of each period t, the sum of expected utility across all bond brokers in any given period is $N_B\alpha_{11}^B\bar{\Xi}\left(A_t^s/N_I\right) y_t = N_I\alpha_{11}\bar{\Xi}\left(A_t^s/N_I\right) y_t$, with $\bar{\Xi}\left(\cdot\right)$ as defined in (262). Hence in a RNE, total welfare (the sum of expected utility across all investors and bond brokers), $V\left(A_t^s, y_t\right)$, satisfies the following recursion

$$
\begin{aligned}
V\left(A_t^s, y_t\right) = N_I \Bigg\{ & \bar{\varepsilon} + \phi^n + \alpha_{11}\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \\
& + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \Bigg] \Bigg\} \frac{A_t^s}{N_I} y_t \\
& + N_I \alpha_{11} \bar{\Xi}\left(A_t^s / N_I\right) y_t - \left[\sigma \left(\phi^n\right)^{\frac{1}{1-\sigma}} + \phi^n \frac{A_t^s}{N_I} \right] N_I y_t \\
& + \beta \mathbb{E}_t V\left[\eta \left(A_t^s + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}} N_I \right), y_{t+1} \right].
\end{aligned}
$$

Substitute the expression for $\bar{\Xi}\left(A_t^s/N_I\right) y_t$ to obtain

$$
\begin{aligned}
V\left(A_t^s, y_t\right) = \Bigg\{ & \bar{\varepsilon} + \alpha_{11} \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \\
& + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \Bigg] \Bigg\} A_t^s y_t \\
& - \sigma \left(\phi^n\right)^{\frac{1}{1-\sigma}} N_I y_t + \beta \mathbb{E}_t V\left[\eta \left(A_t^s + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}} N_I \right), y_{t+1} \right].
\end{aligned} \tag{278}
$$

It is easy to show $V\left(A_t^s, y_t\right) = \left(B A_t^s + C\right) y_t$, where

$$
\begin{aligned}
\left(1 - \bar{\beta}\eta\right) B &= \bar{\varepsilon} + \alpha_{11} \left[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \right] \\
\left(1 - \bar{\beta}\right) C &= \Bigg\{ \frac{\bar{\beta}\eta}{1 - \bar{\beta}\eta} \Bigg\{ \bar{\varepsilon} + \alpha_{11} \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \\
& \quad + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \Bigg] \Bigg\} \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}} - \sigma \left(\phi^n\right)^{\frac{1}{1-\sigma}} \Bigg\} N_I.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(1 - \bar{\beta}\right) V\left(A_t^s, y_t\right) = \frac{1 - \bar{\beta}}{1 - \bar{\beta}\eta} \Bigg\{ & \bar{\varepsilon} + \alpha_{11} \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \\
& + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \Bigg] \Bigg\} A_t^s y_t \\
& + \Bigg\{ \frac{\bar{\beta}\eta}{1 - \bar{\beta}\eta} \Bigg\{ \bar{\varepsilon} + \alpha_{11} \Bigg[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \\
& + \frac{\lambda \phi^n}{\varepsilon^n + (1-\lambda)\phi^n} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right) \Bigg] \Bigg\} \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}} - \sigma \left(\phi^n\right)^{\frac{1}{1-\sigma}} \Bigg\} y_t N_I.
\end{aligned}
$$

In the economy where the period length is Δ, the recursion (278) generalizes to

$$V\left(A_t^s, y_t\right) = \left\{\bar{\varepsilon} + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) \right.\right.$$
$$\left.\left. + \frac{\lambda \Phi^n\left(\Delta\right)}{\varepsilon^n + \left(1-\lambda\right)\Phi^n\left(\Delta\right)} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right)\right]\right\} A_t^s y_t \Delta$$
$$- \sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}} N_I y_t \Delta + \beta \mathbb{E}_t V\left[\eta\left(A_t^s + \left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}} N_I \Delta\right), y_{t+\Delta}\right],$$

where $\sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}} y_t$ is the individual effort rate devoted to investment, and $\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}}$ is the individual investment rate. It is easy to show that the value function for this problem is $V\left(A_t^s, y_t\right) = \left[B\left(\Delta\right) A_t^s + C\left(\Delta\right)\right] y_t$ (proportional to the dividend rate, y_t), with

$$B\left(\Delta\right) = \frac{\Delta}{1-\bar{\beta}\eta}\left\{\bar{\varepsilon} + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon^n} \left(\varepsilon^n - \varepsilon\right) dG\left(\varepsilon\right) + \frac{\lambda \Phi^n\left(\Delta\right)}{\varepsilon^n + \left(1-\lambda\right)\Phi^n\left(\Delta\right)} \int_{\varepsilon^n}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n\right) dG\left(\varepsilon\right)\right]\right\}$$
$$C\left(\Delta\right) = \frac{\Delta}{1-\bar{\beta}}\left[\bar{\beta}\eta B\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}} - \sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}}\right] N_I.$$

Notice that

$$\lim_{\Delta\to 0} B\left(\Delta\right) = \frac{\varphi_1^n}{\rho}$$
$$\lim_{\Delta\to 0} C\left(\Delta\right) = \frac{1}{r-g}\left(\frac{\varphi_1^n}{\varphi^n} - \sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}} N_I.$$

Hence, the limiting expression $\mathcal{V}\left(A_t^s, y_t\right) \equiv \lim_{\Delta\to 0} V\left(A_t^s, y_t\right)$ is as in (276).

(ii) Consider a monetary economy. From (145), the sum of expected discounted utility across all investors at the beginning of period t along a recursive equilibrium where each investor holds A_t^m/N_I dollars and A_t^s/N_I equity shares, is

$$N_I \int V_t^I\left(\frac{A_t^m}{N_I}, \frac{A_t^s}{N_I}, \varepsilon\right) dG\left(\varepsilon\right) = N_I\left(\bar{v}^z \frac{1}{p_t} y_t \frac{A_t^m}{N_I} + \bar{v}^s y_t \frac{A_t^s}{N_I}\right) + N_I \bar{W}_t^I, \tag{279}$$

where $\bar{v}^z$ and $\bar{v}^s$ are given in (265) and (266), and

$$\bar{W}_t^I \equiv T_t + \max_{h_{2t}\in\mathbb{R}_+}\left[\phi_t^s f_t\left(h_{2t}\right) - h_{2t}\right]$$
$$+ \max_{\tilde{\boldsymbol{a}}_{t+1}\in\mathbb{R}_+^2}\left[-\boldsymbol{\phi}_t \tilde{\boldsymbol{a}}_{t+1} + \beta \mathbb{E}_t \int V_{t+1}^I\left(\boldsymbol{a}_{t+1}, \varepsilon\right) dG(\varepsilon)\right].$$

Along a RME path, we have $\phi_t^m A_t^m = Z A_t^s y_t$, $\phi_t^s = \phi^s y_t$, $h_{2t} = g_t\left(\phi_t^s\right) = \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} y_t$, $f_t\left(h_{2t}\right) = x_t\left(\phi_t^s\right) = \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}$, $\tilde{a}_{t+1}^m = A_{t+1}^m/N_I$, $\tilde{a}_{t+1}^s = \left(A_t^s + X_t\right)/N_I$, and $X_t = N_I x_t\left(\phi_t^s\right)$, as described in

Section 5.1. Also, the government budget constraint is $N_I T_t = \phi_t^m \left(A_{t+1}^m - A_t^m\right)$. Hence

$$\begin{aligned}\bar{W}_t^I \equiv & -\left[\sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} + \frac{\left(Z+\phi^s\right) A_t^s}{N_I}\right] y_t \\ & + \beta \mathbb{E}_t \int V_{t+1}^I \left[A_{t+1}^m / N_I, \eta\left(A_t^s / N_I + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon). \end{aligned} \tag{280}$$

Substitute (280) into (279) and use the fact that $p_t = \frac{\left(\varepsilon_{10}^* + \phi^s\right) A_t^m}{Z A_t^s}$ to get

$$\begin{aligned} N_I \int V_t^I \left(\frac{A_t^m}{N_I}, \frac{A_t^s}{N_I}, \varepsilon\right) dG(\varepsilon) = & \left(\frac{\bar{v}^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}^s - \phi^s\right) A_t^s y_t - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t \\ & + \beta \mathbb{E}_t N_I \int V_{t+1}^I \left[\frac{A_{t+1}^m}{N_I}, \eta\left(\frac{A_t^s}{N_I} + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon). \end{aligned}$$

Also, along a recursive equilibrium where each investor holds portfolio $\left(A_t^m / N_I, A_t^s / N_I\right)$ at the beginning of each period t, the sum of expected utility across all bond brokers in any given period is $N_B \alpha_{11}^B \bar{\Xi}\left(Z A_t^s / N_I, A_t^s / N_I\right) y_t = N_I \alpha_{11} \bar{\Xi}\left(Z A_t^s / N_I, A_t^s / N_I\right) y_t$, with $\bar{\Xi}(\cdot, \cdot)$ as defined in (272). Hence in a RME, total welfare (the sum of expected utility across all investors and bond brokers), denoted $V\left(Z A_t^s, A_t^s, y_t\right)$, satisfies the following recursion

$$\begin{aligned} V\left(Z A_t^s, A_t^s, y_t\right) = & \left(\frac{\bar{v}^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}^s - \phi^s\right) A_t^s y_t \\ & + N_I \alpha_{11} \bar{\Xi}\left(Z A_t^s / N_I, A_t^s / N_I\right) y_t - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t \\ & + \beta \mathbb{E}_t V\left[Z \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), y_{t+1}\right]. \end{aligned}$$

Substitute the expression for $\bar{\Xi}\left(Z A_t^s / N_I, A_t^s / N_I\right)$ to obtain

$$\begin{aligned} V\left(Z A_t^s, A_t^s, y_t\right) = & \left(\frac{\bar{v}_1^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}_1^s - \phi^s\right) A_t^s y_t \\ & - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t + \beta \mathbb{E}_t V\left[Z \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), y_{t+1}\right], \end{aligned}$$

where

$$\begin{aligned} \bar{v}_1^z \equiv & \varepsilon_{10}^* + \phi^s + \alpha_{10} \int_{\varepsilon_{10}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{10}^*\right) dG(\varepsilon) \\ & + \alpha_{11}\left[\varepsilon_{11}^* - \varepsilon_{10}^* + \frac{\varepsilon_{11}^* + \phi^s}{\varepsilon_{11}^* + (1-\lambda) \phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11}^*\right) dG(\varepsilon)\right] \\ \bar{v}_1^s \equiv & \bar{\varepsilon} + \phi^s + \alpha_{10} \int_{\varepsilon_L}^{\varepsilon_{10}^*} \left(\varepsilon_{10}^* - \varepsilon\right) dG(\varepsilon) \\ & + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon_{11}^*} \left(\varepsilon_{11}^* - \varepsilon\right) dG(\varepsilon) + \frac{\lambda \phi^s}{\varepsilon_{11}^* + (1-\lambda) \phi^s} \int_{\varepsilon_{11}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{11}^*\right) dG(\varepsilon)\right]. \end{aligned}$$

It is easy to show $V\left(ZA_t^s, A_t^s, y_t\right) = \left(BA_t^s + C\right) y_t$, where

$$\left(1-\bar{\beta}\eta\right) B = \frac{\bar{v}_1^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}_1^s - \phi^s$$

$$\left(1-\bar{\beta}\right) C = \frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\left[\frac{\bar{v}_1^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}_1^s - \phi^s\right]\left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I.$$

Hence

$$\begin{aligned}\left(1-\bar{\beta}\right) V\left(ZA_t^s, A_t^s, y_t\right) = & \frac{1-\bar{\beta}}{1-\bar{\beta}\eta}\left[\frac{\bar{v}_1^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}_1^s - \phi^s\right] A_t^s y_t \\ & + \frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\left[\frac{\bar{v}_1^z - \varepsilon_{10}^* - \phi^s}{\varepsilon_{10}^* + \phi^s} Z + \bar{v}_1^s - \phi^s\right]\left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I y_t \\ & - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t.\end{aligned}$$

In the discrete-time economy where the period length is Δ, this value function generalizes to

$$\begin{aligned}& \frac{(r-g)\Delta}{1+r\Delta} V\left(\mathrm{Z}(\Delta)\Delta A_t^s, A_t^s, y_t\right) \\ & = \left\{\left[\frac{\bar{v}_1^z(\Delta) - \varepsilon_{10}^* - \Phi^s(\Delta)}{\varepsilon_{10}^* + \Phi^s(\Delta)} \mathrm{Z}(\Delta) + \bar{v}_1^s(\Delta) - \Phi^s(\Delta)\right] y_t\Delta\right\}\left\{\frac{\frac{(r-g)\Delta}{1+r\Delta}}{\frac{r+\delta-g+g\delta\Delta}{1+r\Delta}\Delta} A_t^s\right. \\ & \left. + \frac{(1+g\Delta)(1-\delta\Delta)}{(r+\delta-g+g\delta\Delta)\Delta}\left[\Phi^s(\Delta)\Delta\right]^{\frac{\sigma}{1-\sigma}}\Delta N_I\right\} \\ & - \sigma\left[\Phi^s(\Delta)\Delta\right]^{\frac{1}{1-\sigma}} N_I y_t \Delta,\end{aligned}$$

with

$$\begin{aligned}\bar{v}_1^z(\Delta) \equiv & \varepsilon_{10}^* + \Phi^s(\Delta) + \alpha_{10}\int_{\varepsilon_{10}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{10}^*\right) dG(\varepsilon) \\ & + \alpha_{11}\left[\varepsilon_{11}^* - \varepsilon_{10}^* + \frac{\varepsilon_{11}^* + \Phi^s(\Delta)}{\varepsilon_{11}^* + (1-\lambda)\Phi^s(\Delta)}\int_{\varepsilon_{11}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{11}^*\right) dG(\varepsilon)\right] \\ \bar{v}_1^s(\Delta) \equiv & \bar{\varepsilon} + \Phi^s(\Delta) + \alpha_{10}\int_{\varepsilon_L}^{\varepsilon_{10}^*}\left(\varepsilon_{10}^* - \varepsilon\right) dG(\varepsilon) \\ & + \alpha_{11}\left[\int_{\varepsilon_L}^{\varepsilon_{11}^*}\left(\varepsilon_{11}^* - \varepsilon\right) dG(\varepsilon) + \frac{\lambda\Phi^s(\Delta)}{\varepsilon_{11}^* + (1-\lambda)\Phi^s(\Delta)}\int_{\varepsilon_{11}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{11}^*\right) dG(\varepsilon)\right].\end{aligned}$$

As usual, $\sigma\left(\Phi^s(\Delta)\Delta\right)^{\frac{1}{1-\sigma}} y_t$ is the individual effort *rate* devoted to investment (so the effort accumulated over a period of length Δ is $\sigma\left(\Phi^s(\Delta)\Delta\right)^{\frac{1}{1-\sigma}} y_t\Delta$), and $\left(\Phi^s(\Delta)\Delta\right)^{\frac{\sigma}{1-\sigma}}$ is the individual investment *rate* (so $\left(\Phi^s(\Delta)\Delta\right)^{\frac{\sigma}{1-\sigma}}\Delta$ is the investment accumulated over a period of length Δ). Notice that$\lim_{\Delta\to 0}\left[\bar{v}_1^z(\Delta)-\varepsilon_{10}^*-\Phi^s(\Delta)\right]=u_1^z$, $\lim_{\Delta\to 0}\left[\bar{v}_1^s(\Delta)-\Phi^s(\Delta)\right]=\varphi_1$ (with u_1^z and φ_1 as defined in part (ii) of Lemma 27), and $\lim_{\Delta\to 0}\frac{\mathrm{Z}(\Delta)}{\varepsilon_{10}^*+\Phi^s(\Delta)}=\frac{\mathcal{Z}}{\varphi}$, so taking the limit as $\Delta\to 0$ and letting $\mathcal{V}(\mathcal{Z},A_t^s,y_t)\equiv\lim_{\Delta\to 0}V(\mathrm{Z}(\Delta)\,\Delta A_t^s,A_t^s,y_t)$, we arrive at (277). ■

Proof of Corollary 1. The fact that $\mathcal{V}^n(y_t)\leq\mathcal{V}^m(\mathcal{Z},y_t)$, with "=" only if $\iota=\bar{\iota}(\lambda)$ is immediate from part (i) of Proposition 3 and the fact that $0\leq\mathcal{Z}$. To show $\mathcal{V}^m(\mathcal{Z},y_t)\leq\mathcal{W}^*(y_t)$, use (107) to rewrite $\mathcal{V}^m(\mathcal{Z},y_t)$ as follows

$$\mathcal{V}^m(\mathcal{Z},y_t)=\frac{1}{r-g}\left[\bar{\varepsilon}+(\alpha_{10}+\alpha_{11})(\varepsilon_{10}^*-\bar{\varepsilon})+\left(1+\frac{\mathcal{Z}}{\varphi}\right)u_1^z\right]A^s y_t.$$

Then substitute (105) to get

$$\begin{aligned}\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z},y_t)&=\bar{\varepsilon}+(\alpha_{10}+\alpha_{11})(\varepsilon_{10}^*-\bar{\varepsilon})\\&+\left(1+\frac{\mathcal{Z}}{\varphi}\right)\left\{\alpha_{10}\int_{\varepsilon_{10}^*}^{\varepsilon_H}(\varepsilon-\varepsilon_{10}^*)\,dG(\varepsilon)\right.\\&\left.+\alpha_{11}\left[\varepsilon_{11}^*-\varepsilon_{10}^*+\frac{1}{1-\lambda}\int_{\varepsilon_{11}^*}^{\varepsilon_H}(\varepsilon-\varepsilon_{11}^*)\,dG(\varepsilon)\right]\right\}.\end{aligned}$$

Next we consider two cases. Case 1: If $\hat{\iota}(\lambda)<\iota<\bar{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by (46), and $\varepsilon_{11}^*=\varepsilon^n$, and therefore

$$\begin{aligned}\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z},y_t)&=\bar{\varepsilon}+(\alpha_{10}+\alpha_{11})\left\{\frac{\alpha_{10}}{[1-G(\varepsilon_{10}^*)]\,\alpha_{10}+\alpha_{11}}\int_{\varepsilon_{10}^*}^{\varepsilon_H}(\varepsilon-\bar{\varepsilon})\,dG(\varepsilon)\right.\\&\left.+\frac{\alpha_{11}}{[1-G(\varepsilon_{10}^*)]\,\alpha_{10}+\alpha_{11}}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]\right\}\\&\leq\bar{\varepsilon}+(\alpha_{10}+\alpha_{11})\left\{\frac{[1-G(\varepsilon_{10}^*)]\,\alpha_{10}}{[1-G(\varepsilon_{10}^*)]\,\alpha_{10}+\alpha_{11}}(\varepsilon_H-\bar{\varepsilon})\right.\\&\left.+\frac{\alpha_{11}}{[1-G(\varepsilon_{10}^*)]\,\alpha_{10}+\alpha_{11}}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]\right\}\\&<\psi=\frac{r-g}{A^s y_t}\mathcal{W}^*(y_t).\end{aligned}$$

The last inequality follows from (179) and (180) that imply

$$\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)<\varepsilon_H-\bar{\varepsilon}. \tag{281}$$

Case 2: If $0 < \iota \leq \hat{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by the expression in part (ii) of Proposition 2, and $\varepsilon^*_{10} = \varepsilon^*_{11} = \varepsilon^*$, and therefore

$$\begin{aligned}\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z}, y_t) &= \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left[\varepsilon^* - \bar{\varepsilon} + \frac{1}{1-G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon)\right] \\ &\leq \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left[\varepsilon^* - \bar{\varepsilon} + \frac{1}{1-G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon_H - \varepsilon^*)\,dG(\varepsilon)\right] \\ &= \psi = \frac{r-g}{A^s y_t}\mathcal{W}^*(y_t),\end{aligned}$$

where the inequality is strict unless $\iota = 0$ (which implies $\varepsilon^* = \varepsilon_H$). ■

Proof of Corollary 3. First, note that

$$\begin{aligned}(r-g)\left[\mathcal{V}^m(\mathcal{Z}, A^s_t, y_t) - \mathcal{V}^n(A^s_t, y_t)\right]\frac{1}{y_t} = &\frac{r-g}{\rho}\left(u^z_1\frac{\mathcal{Z}}{\varphi} + \varphi_1 - \varphi^n_1\right)A^s_t \\ &+ \left(\frac{u^z_1\frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I \\ &- \left(\frac{\varphi^n_1}{\varphi^n} - \sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}} N_I. \qquad (282)\end{aligned}$$

The first term is strictly positive unless $\iota = \bar{\iota}(\lambda)$ (because $\varphi^n_1 \leq \varphi_1$ by Proposition 3, and $0 \leq \mathcal{Z}$, and both inequalities are strict unless $\iota = \bar{\iota}(\lambda)$). Hence to show $\mathcal{V}^n(A^s_t, y_t) \leq \mathcal{V}^m(\mathcal{Z}, A^s_t, y_t)$, it is sufficient to show that the sum of the last two terms in (282) is nonnegative (and positive unless $\iota = \bar{\iota}(\lambda)$ and $\theta = 1$). Define

$$\Omega(x, y) \equiv \left(\frac{y}{x} - \sigma\right)\left(\frac{x}{\rho}\right)^{\frac{1}{1-\sigma}}.$$

Notice

$$\frac{\partial}{\partial y}\Omega(x, y) = \frac{1}{x}\left(\frac{x}{\rho}\right)^{\frac{1}{1-\sigma}} > 0 \qquad (283)$$

$$\frac{\partial}{\partial x}\Omega(x, y) = \left(\frac{y}{x} - 1\right)\frac{1}{\rho}\frac{\sigma}{1-\sigma}\left(\frac{y}{x}\right)^{\frac{\sigma}{1-\sigma}} > 0 \text{ if and only if } x < y. \qquad (284)$$

Then

$$\begin{aligned}\left(\frac{u^z_1\frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} - \left(\frac{\varphi^n_1}{\varphi^n} - \sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}} &\geq \Omega(\varphi, \varphi_1) - \Omega(\varphi^n, \varphi^n_1) \\ &\geq \Omega(\varphi, \varphi_1) - \Omega(\varphi^n, \varphi_1) \\ &\geq \Omega(\varphi^n, \varphi_1) - \Omega(\varphi^n, \varphi_1) = 0.\end{aligned}$$

The second inequality follows from (283) and the fact that $\varphi_1^n \leq \varphi_1$. The third inequality follows from (284) and the fact that $\varphi^n \leq \varphi \leq \varphi_1$. Thus, $\mathcal{V}^n(A_t^s, y_t) \leq \mathcal{V}^m(\mathcal{Z}, A_t^s, y_t)$, with equality only if $\iota = \bar{\iota}(\lambda)$ and $\theta = 1$ (since in this case, $\mathcal{Z} = 0$ and $\varphi = \varphi_1 = \varphi^n = \varphi_1^n$).

To show that $\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$, proceed as follows. From (119) and (121),

$$\begin{aligned}
&(r-g)\left[\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) - \mathcal{W}^*(A_t^s, y_t)\right]\frac{1}{y_t} \\
&= \frac{r-g}{\rho}\left(u_1^z\frac{\mathcal{Z}}{\varphi} + \varphi_1 - \psi\right)A_t^s \\
&+ \left[\left(\frac{u_1^z\frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I - (1-\sigma)\left(\frac{\psi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right]. \qquad (285)
\end{aligned}$$

We first show the first term in (285) is nonpositive (strictly negative unless $\iota = 0$). To this end, we consider two cases in turn. First, if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by (46), and $\varepsilon_{11}^* = \varepsilon^n$, and therefore

$$\begin{aligned}
u_1^z\frac{\mathcal{Z}}{\varphi} + \varphi_1 &= \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left\{\frac{\alpha_{10}}{\left[1 - G(\varepsilon_{10}^*)\right]\alpha_{10} + \alpha_{11}}\int_{\varepsilon_{10}^*}^{\varepsilon_H}(\varepsilon - \bar{\varepsilon})\,dG(\varepsilon)\right. \\
&\left.+ \frac{\alpha_{11}}{\left[1 - G(\varepsilon_{10}^*)\right]\alpha_{10} + \alpha_{11}}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right]\right\} \\
&\leq \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left\{\frac{\left[1 - G(\varepsilon_{10}^*)\right]\alpha_{10}}{\left[1 - G(\varepsilon_{10}^*)\right]\alpha_{10} + \alpha_{11}}(\varepsilon_H - \bar{\varepsilon})\right. \\
&\left.+ \frac{\alpha_{11}}{\left[1 - G(\varepsilon_{10}^*)\right]\alpha_{10} + \alpha_{11}}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right]\right\} \\
&< \psi,
\end{aligned}$$

where the last inequality follows from (179) and (180) that imply (281). Hence the first term in (285) is negative if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$. Second, if $0 < \iota \leq \hat{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by the expression in part (*ii*) of Proposition 2, and $\varepsilon_{10}^* = \varepsilon_{11}^* = \varepsilon^*$, and therefore

$$\begin{aligned}
u_1^z\frac{\mathcal{Z}}{\varphi} + \varphi_1 &= \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left[\varepsilon^* - \bar{\varepsilon} + \frac{1}{1 - G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon)\right] \\
&\leq \bar{\varepsilon} + (\alpha_{10} + \alpha_{11})\left[\varepsilon^* - \bar{\varepsilon} + \frac{1}{1 - G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon_H - \varepsilon^*)\,dG(\varepsilon)\right] = \psi,
\end{aligned}$$

where the inequality is strict unless $\iota = 0$ (which implies $\varepsilon^* = \varepsilon_H$). Hence regardless of whether $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$ or $0 < \iota \leq \hat{\iota}(\lambda)$, we have $u_1^z\frac{\mathcal{Z}}{\varphi} + \varphi_1 \leq \psi$ (with "=" only if $\iota = 0$), so to show

$\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$ it is sufficient to show the second term in (285) is nonpositive. This can be shown as follows

$$\begin{aligned}\left(\frac{u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1}{\varphi}-\sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}}-(1-\sigma)\left(\frac{\psi}{\rho}\right)^{\frac{1}{1-\sigma}} &= \Omega\left(\varphi, u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1\right)-\Omega(\psi,\psi)\\ &\leq \Omega(\varphi,\psi)-\Omega(\psi,\psi)\\ &\leq \Omega(\psi,\psi)-\Omega(\psi,\psi)=0.\end{aligned}$$

The first inequality follows from (283) and $u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1 \leq \psi$. The second inequality follows from (284) and $\varphi \leq \psi$. Thus, $\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$, with "=" only if $\iota = 0$ (since in this case $u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1=\varphi=\psi$). ■

A.12 Effects of monetary policy

Proof of Proposition 15. (i) The condition that characterizes ε_{10}^* in part (i) of Proposition 2 can be written as

$$\begin{aligned}\varphi\iota = {} & \alpha_{11}\theta\left(\varepsilon^n-\varepsilon_{10}^*\right)+\left[\alpha_{10}+\alpha_{11}(1-\theta)\right]\int_{\varepsilon_{10}^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{10}^*\right)dG(\varepsilon)\\ &+\alpha_{11}\theta\frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG(\varepsilon).\end{aligned}$$

Totally differentiate this condition with respect to ι to get

$$\varphi+\iota\frac{d\varphi}{d\iota} = -\left\{\alpha_{11}\theta+\left[\alpha_{10}+\alpha_{11}(1-\theta)\right]\left[1-G\left(\varepsilon_{10}^*\right)\right]\right\}\frac{d\varepsilon_{10}^*}{d\iota}. \tag{286}$$

Totally differentiate (45) with respect to ι to get

$$\frac{d\varphi}{d\iota} = \left[\alpha_{10}+\alpha_{11}(1-\theta)\right]G\left(\varepsilon_{10}^*\right)\frac{d\varepsilon_{10}^*}{d\iota}. \tag{287}$$

Together, (286) and (287) imply

$$-\frac{d\varphi}{d\iota}\frac{\iota}{\varphi} = \frac{\iota}{\iota+\frac{\alpha_{11}\theta+\left[\alpha_{10}+\alpha_{11}(1-\theta)\right]\left[1-G\left(\varepsilon_{10}^*\right)\right]}{\left[\alpha_{10}+\alpha_{11}(1-\theta)\right]G\left(\varepsilon_{10}^*\right)}}.$$

(ii) The condition that characterizes ε^* in part (ii) of Proposition 2 can be written as

$$\varphi\iota = \left[\alpha_{10}+\alpha_{11}\left(1+\theta\frac{\lambda}{1-\lambda}\right)\right]\int_{\varepsilon^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon^*\right)dG(\varepsilon).$$

Totally differentiate this condition to get

$$\varphi + \iota \frac{d\varphi}{d\iota} = -\left[\alpha_{10} + \alpha_{11}\left(1 + \theta \frac{\lambda}{1-\lambda}\right)\right]\left[1 - G\left(\varepsilon^*\right)\right] \frac{d\varepsilon^*}{d\iota}. \tag{288}$$

Totally differentiate the expression for φ in part (ii) of Proposition 2 to get

$$\frac{d\varphi}{d\iota} = \left\{\left(\alpha_{10} + \alpha_{11}\right) G\left(\varepsilon^*\right) - \alpha_{11}\theta \frac{\lambda}{1-\lambda}\left[1 - G\left(\varepsilon^*\right)\right]\right\} \frac{d\varepsilon^*}{d\iota}. \tag{289}$$

Combine (288) and (289) to get

$$\frac{d\varphi}{d\iota}\frac{\iota}{\varphi} = -\frac{\iota}{\iota + \frac{\left[\alpha_{10}+\alpha_{11}\left(1+\theta\frac{\lambda}{1-\lambda}\right)\right]\left[1-G(\varepsilon^*)\right]}{(\alpha_{10}+\alpha_{11})G(\varepsilon^*)-\alpha_{11}\theta\frac{\lambda}{1-\lambda}\left[1-G(\varepsilon^*)\right]}}.$$

■

B Quantitative robustness

In this section we assess the robustness of the quantitative results of Section 8 to alternative calibration strategies. In our baseline, the parameters α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following three facts: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate, as in the high-frequency empirical estimates in Lagos and Zhang (2019); (b) transaction velocity of money is 25 per day, which is the average daily number of times a dollar turns over in CHIPS (Clearing House Interbank Payments System); and (c) the median spread on margin loans is about 2.3%, which is the current spread (over the fed funds rate) that a typical prime broker charges a large investor. This procedure delivers $\alpha = .0406$, $\theta = .1612$, and $\Sigma_\varepsilon = 2.0784$. Below, we report results for three alternative calibrations that consider alternative target values for the spread on margin loans and/or velocity.

In the first alternative calibration, denoted (AC1), α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following targets: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is 25 per day; and (c) the median spread on margin loans is about 1.20%. This procedure delivers $\alpha = .0389$, $\theta = .2979$, and $\Sigma_\varepsilon = 2.3653$. Figure 12 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0,1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\lambda = .75$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 13 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0,1] \times \{.10, .30, .70, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\theta = .2979$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 14 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0,1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 15, 16, and 17 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 15 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as $\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 16 and 17 tell a similar story. Figure 16, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

In the second alternative calibration, denoted (AC2), α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following targets: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is about 6 per day; and (c) the median spread on margin loans is about 25 basis points. This procedure delivers $\alpha = .0966$, $\theta = .8337$, and $\Sigma_\varepsilon = 2.6429$. Figure 18 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0,1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0966$ and $\lambda = .75$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 19 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0,1] \times \{.10, .25, .83, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0966$ and $\theta = .8337$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 20 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0,1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that

$\mathcal{S} = 11$ for $\alpha = .0966$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 21, 22, and 23 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 21 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as $\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 22 and 23 tell a similar story. Figure 22, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

In the third alternative calibration, denoted (AC3), we set $\alpha = 0$, and λ, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is about 25 per day; and (c) the median spread on margin loans is about 25 basis points. This procedure delivers $\lambda = .9159$, $\theta = .8080$, and $\Sigma_\varepsilon = 3.0886$. Figure 24 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0, 1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\lambda = .9159$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 25 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0, 1] \times \{.10, .25, .80, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\theta = .8080$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 26 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0, 1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 27, 28, and 29 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 27 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as $\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 28 and 29 tell a similar story. Figure

28, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

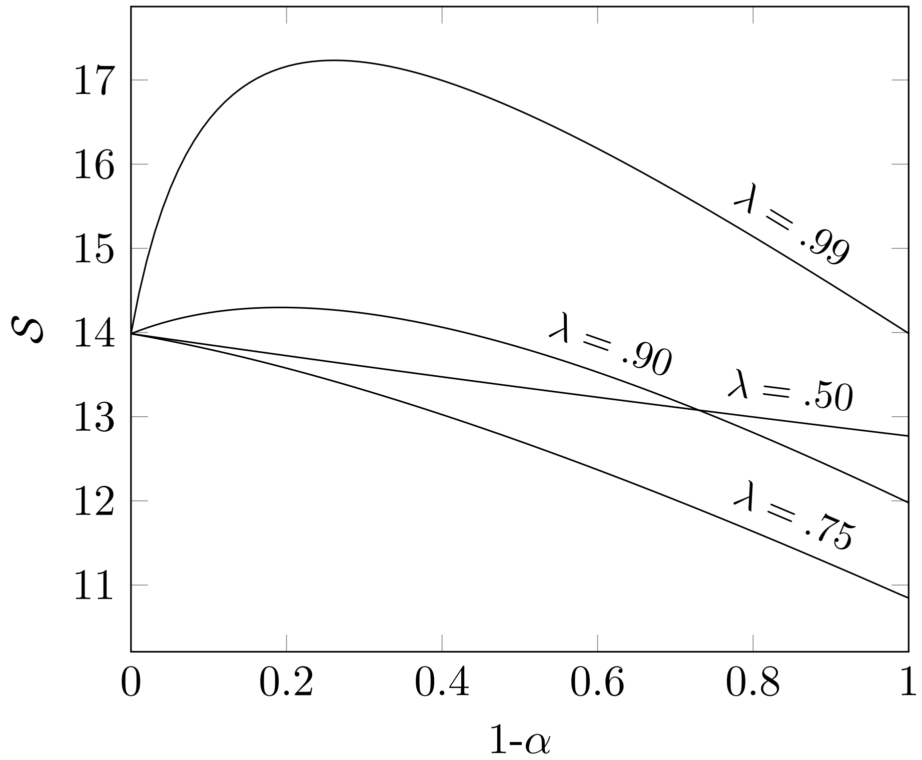


Figure 12: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1 - \alpha$.

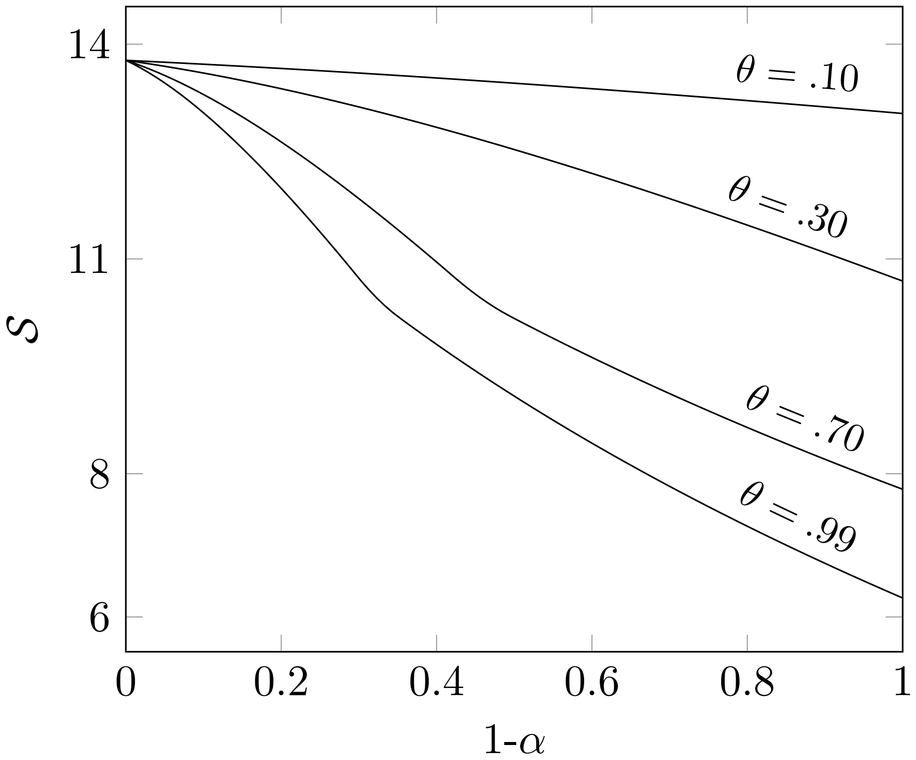


Figure 13: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1 - \theta$, and access to credit, $1 - \alpha$.

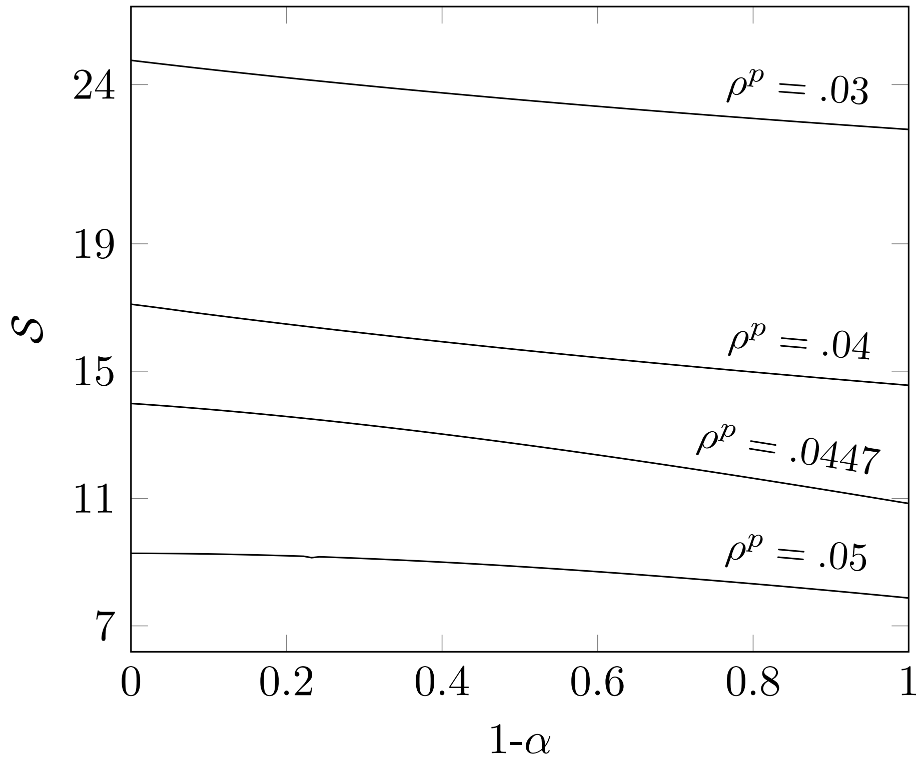


Figure 14: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1 - \alpha$.

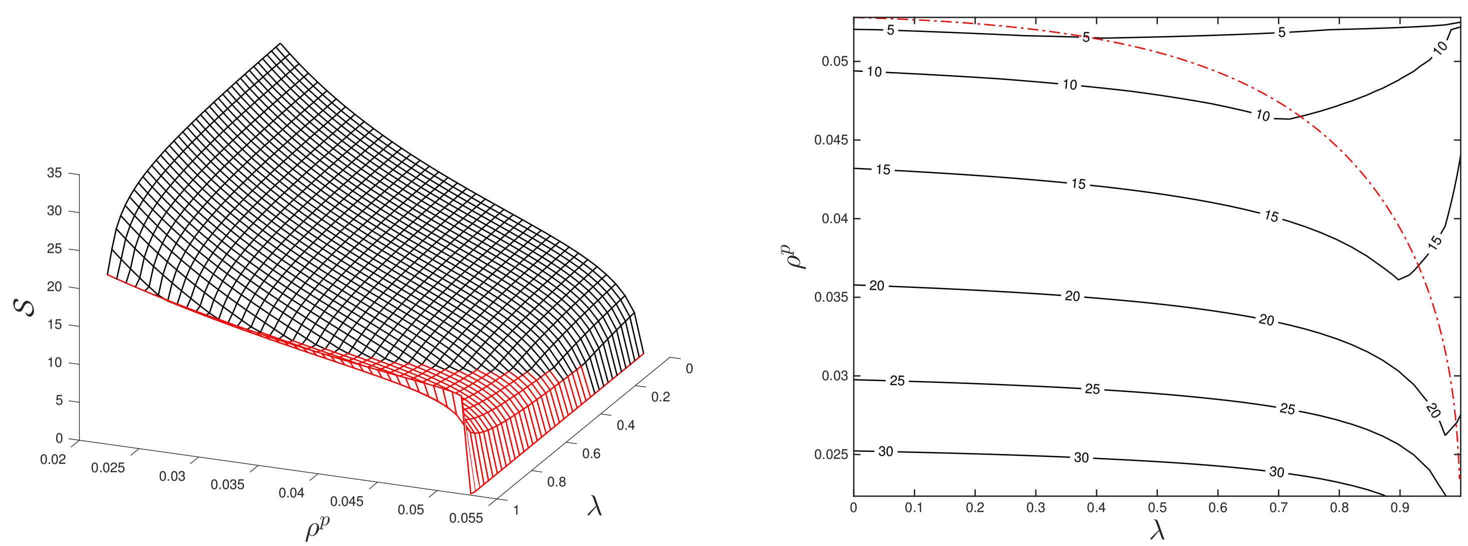


Figure 15: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

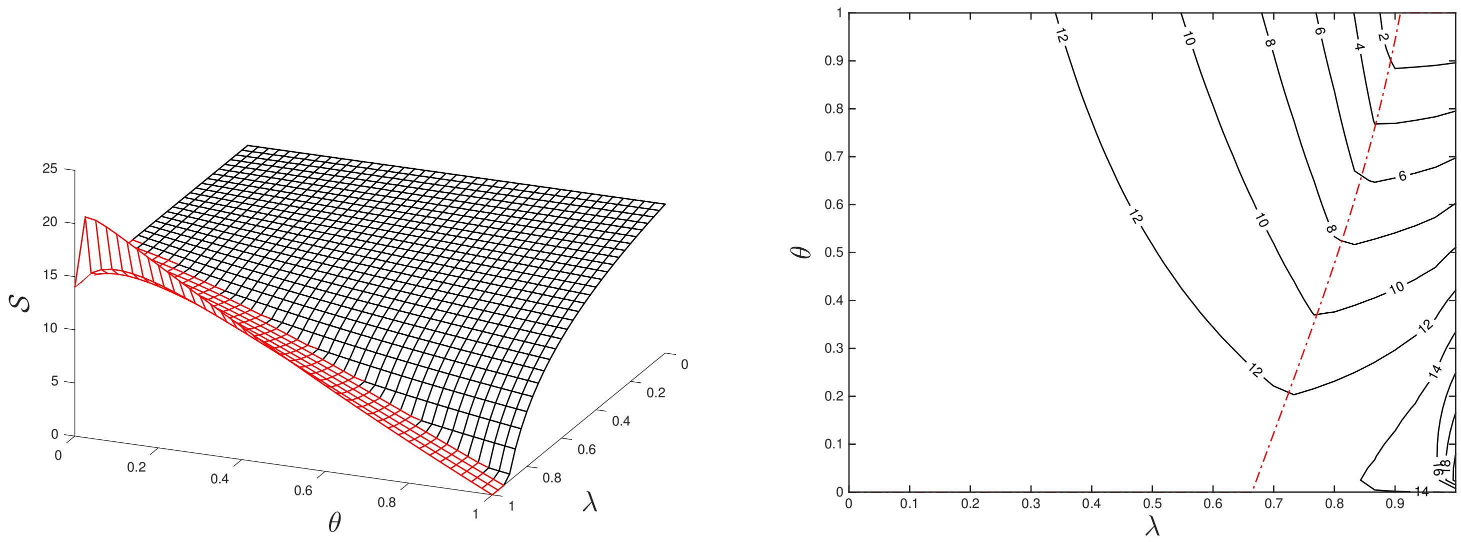


Figure 16: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).

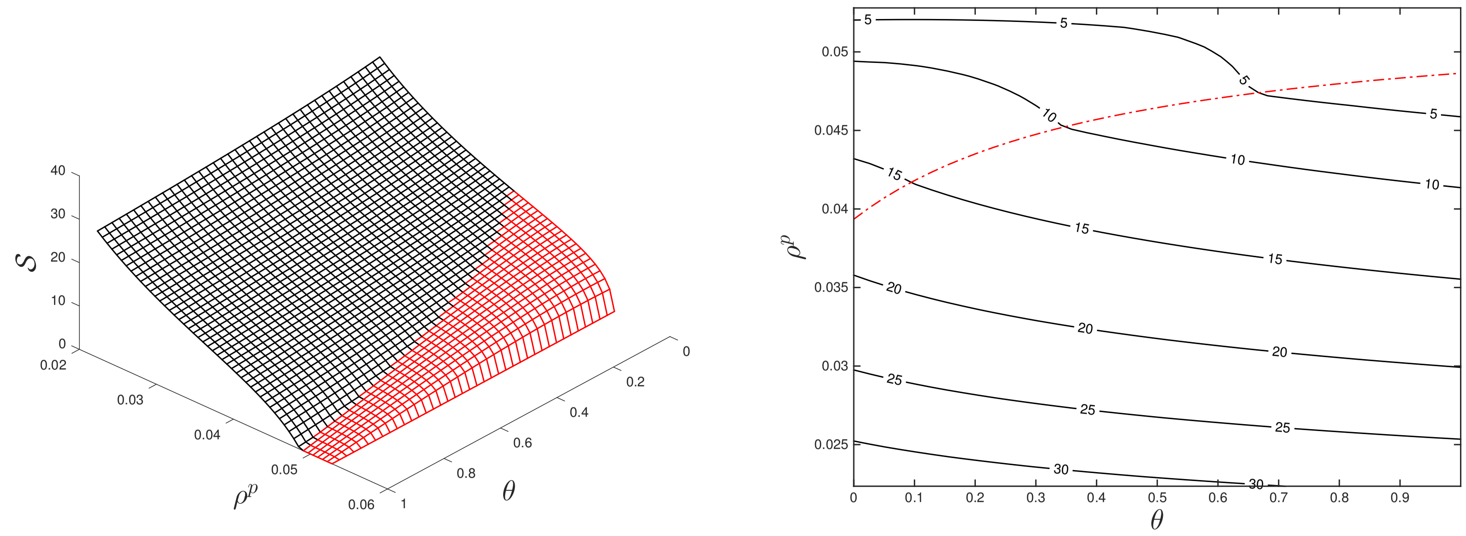


Figure 17: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

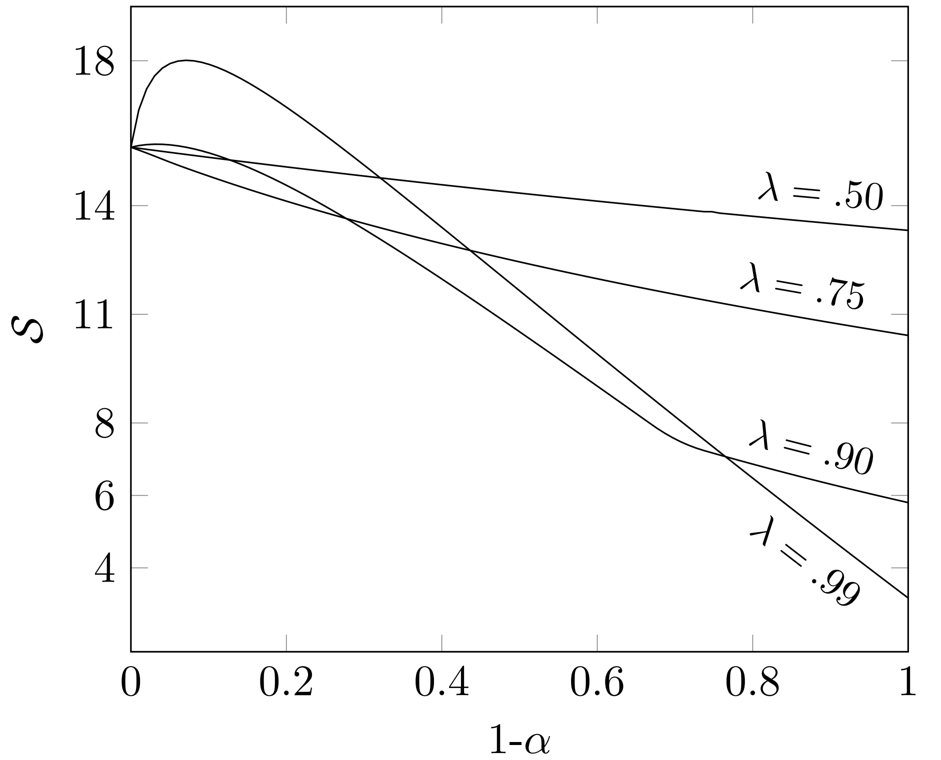


Figure 18: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1 - \alpha$.

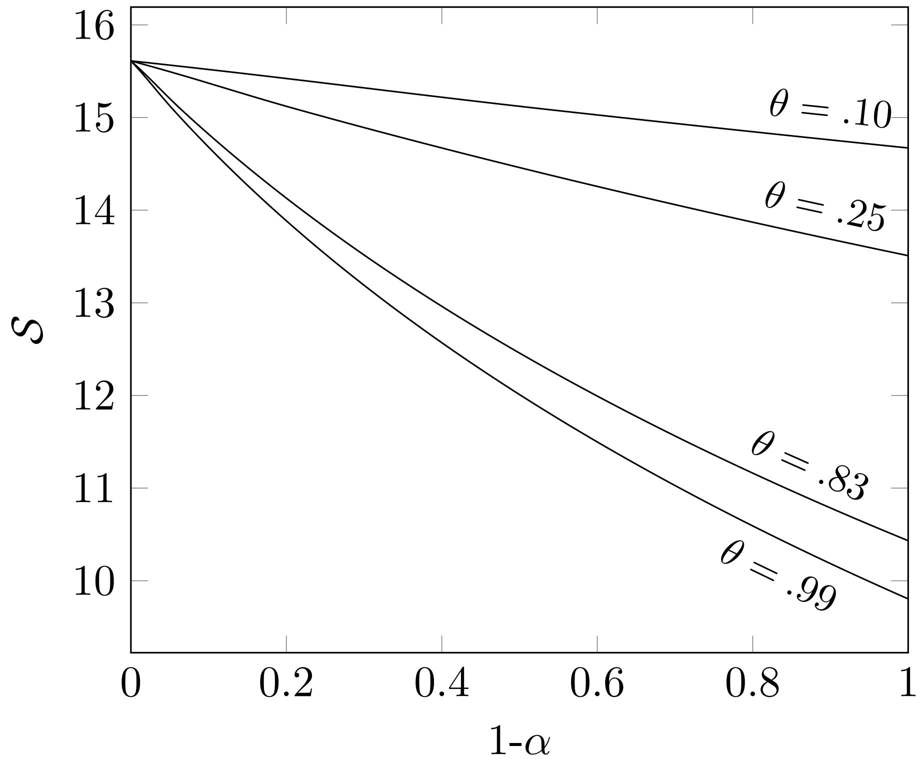


Figure 19: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1-\theta$, and access to credit, $1-\alpha$.

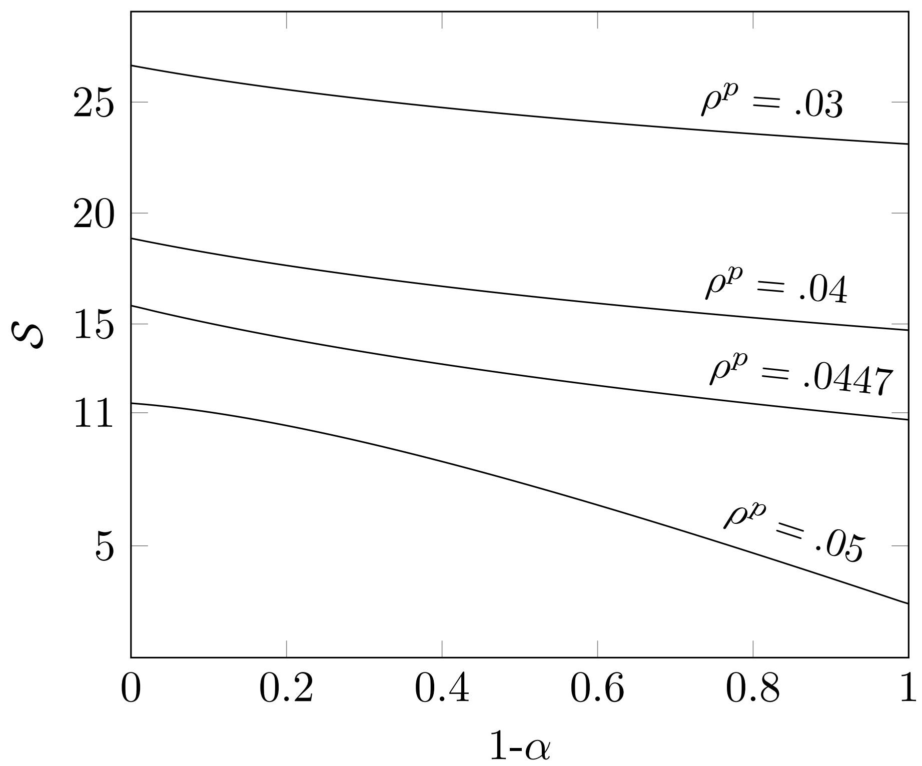


Figure 20: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1 - \alpha$.

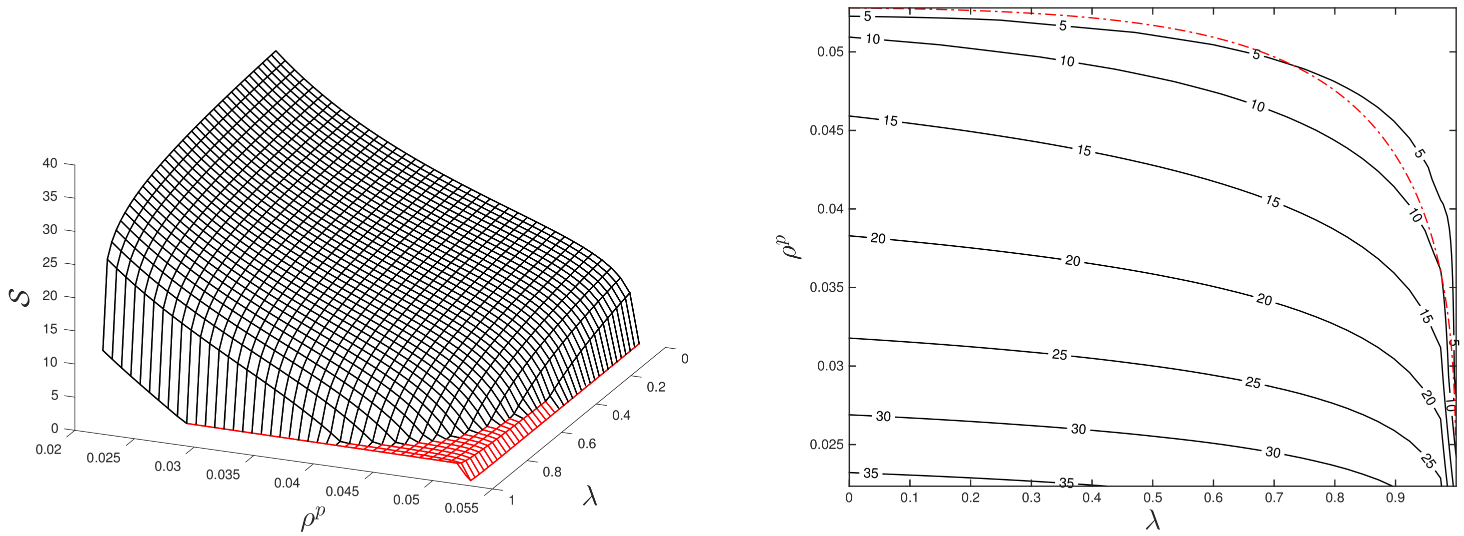


Figure 21: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

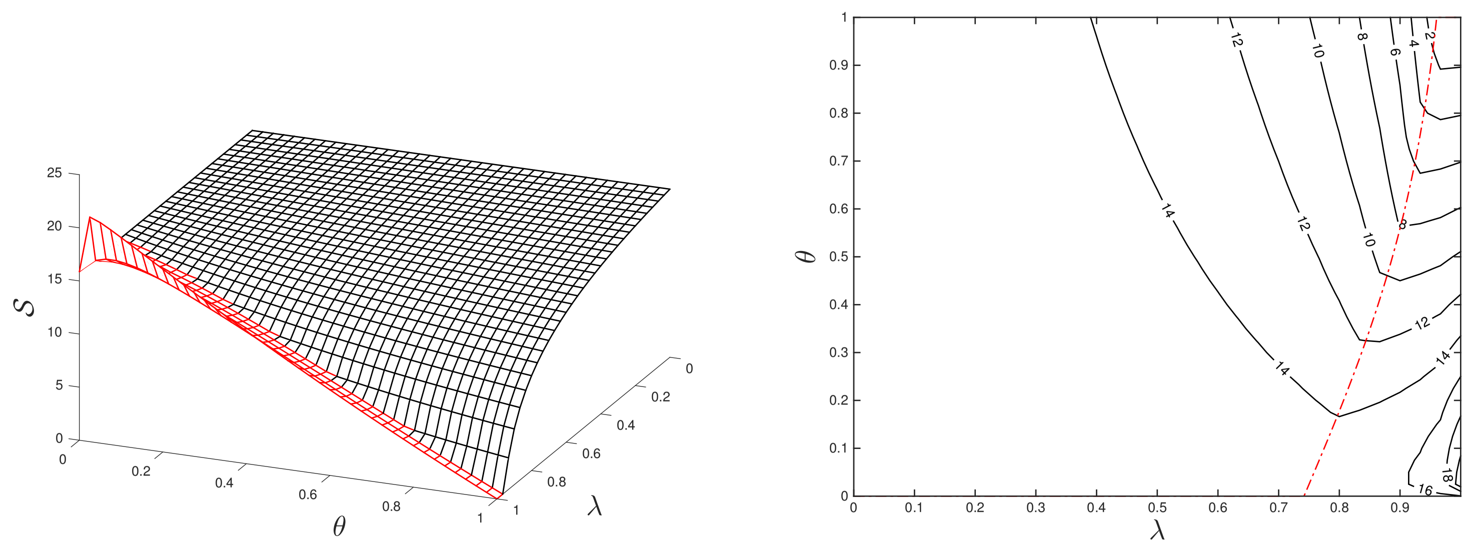


Figure 22: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).

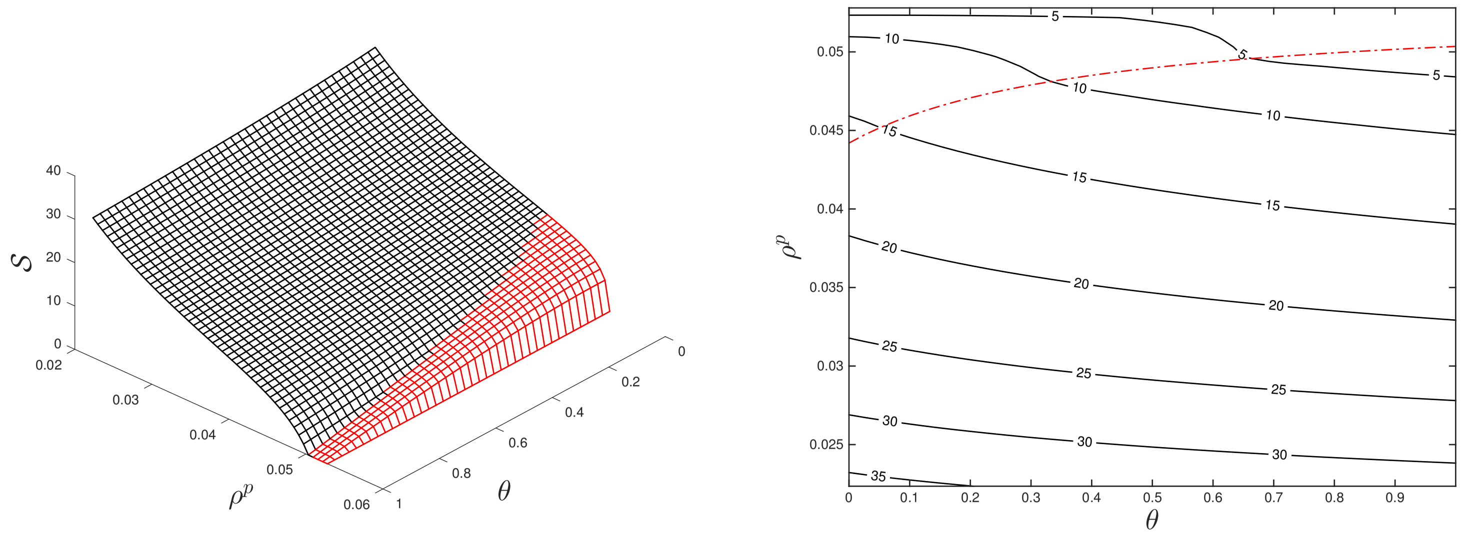


Figure 23: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

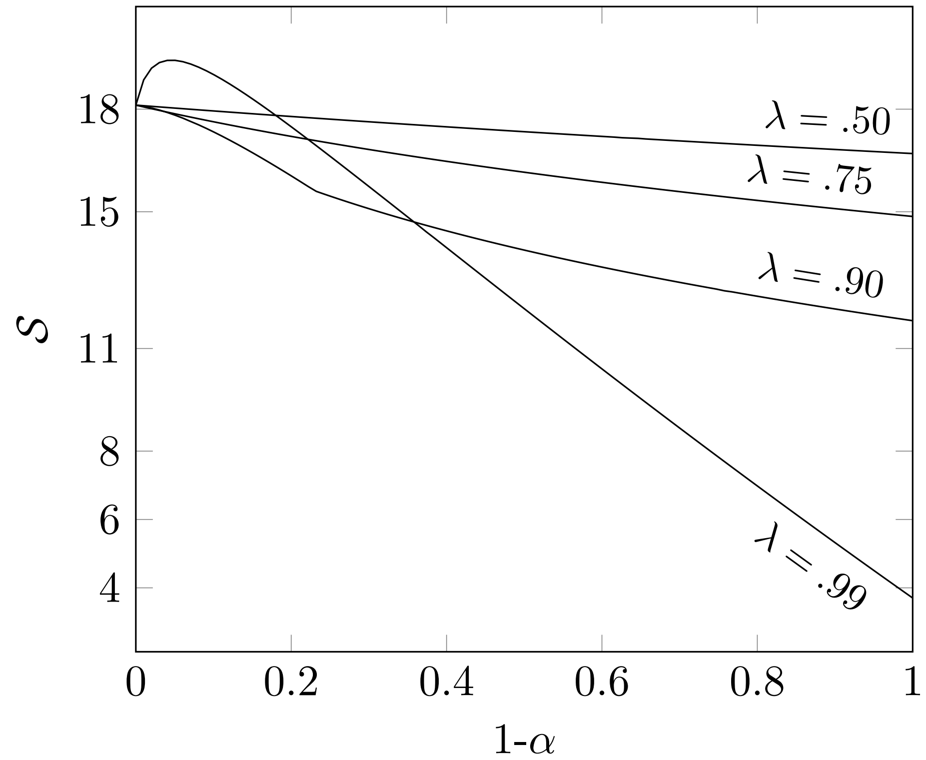


Figure 24: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1 - \alpha$.

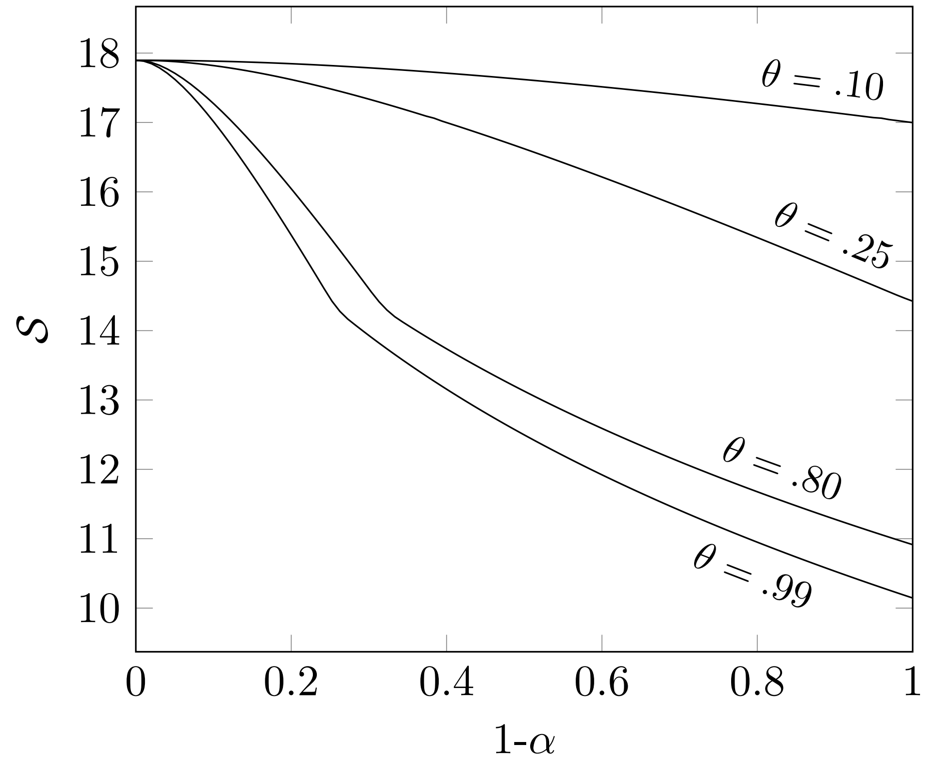


Figure 25: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1 - \theta$, and access to credit, $1 - \alpha$.

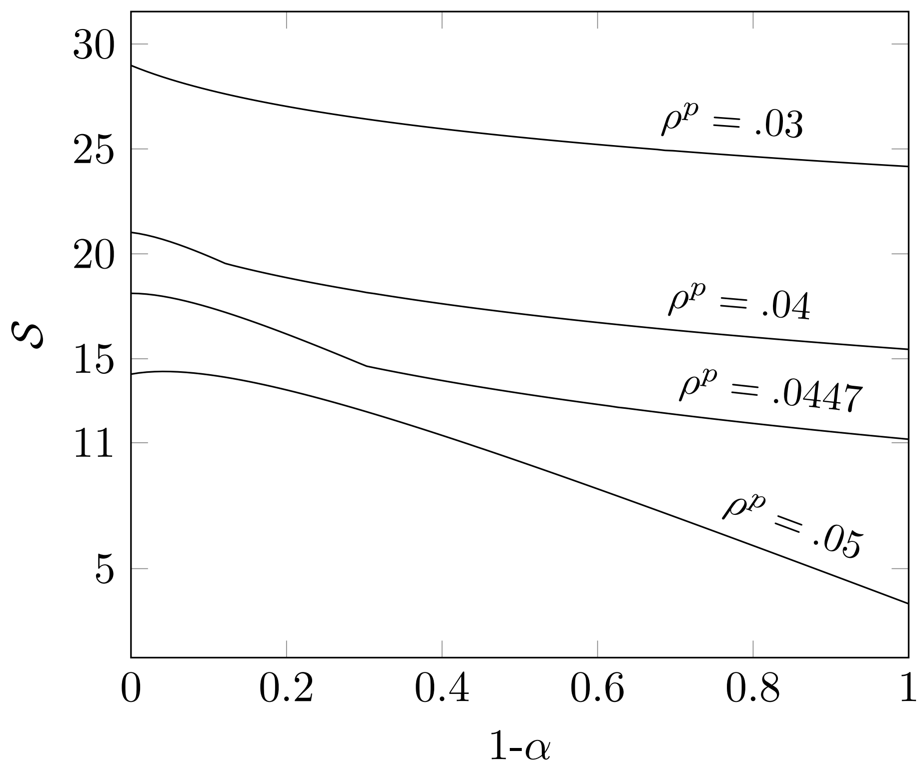


Figure 26: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1-\alpha$.

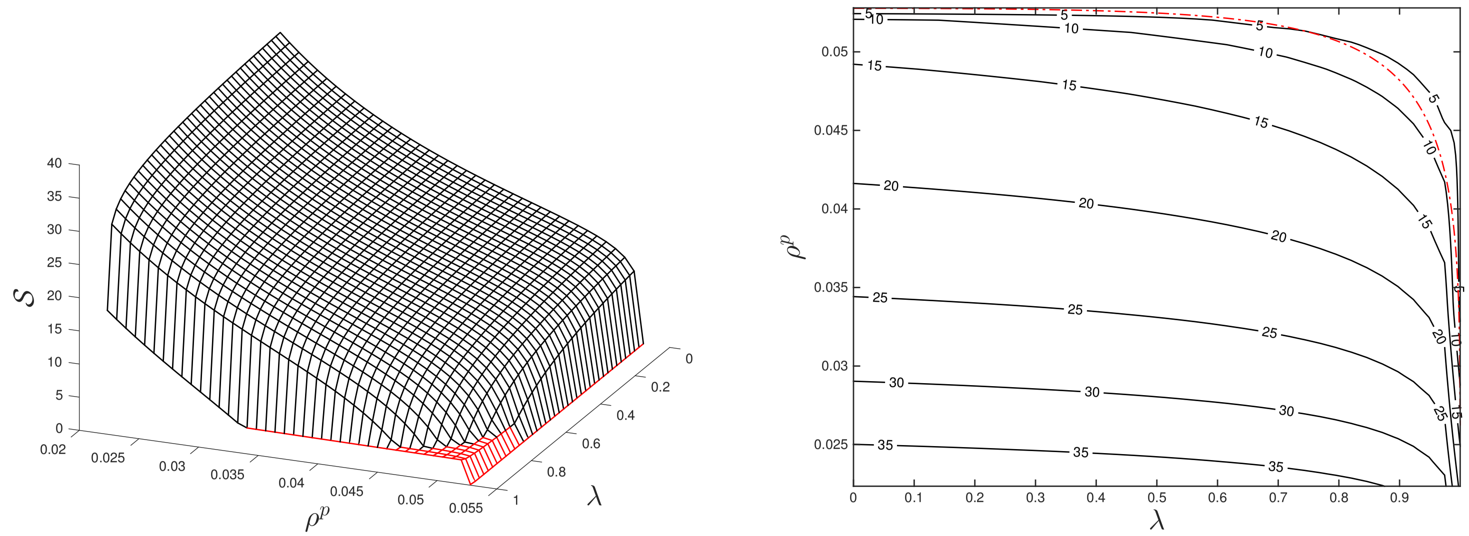


Figure 27: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

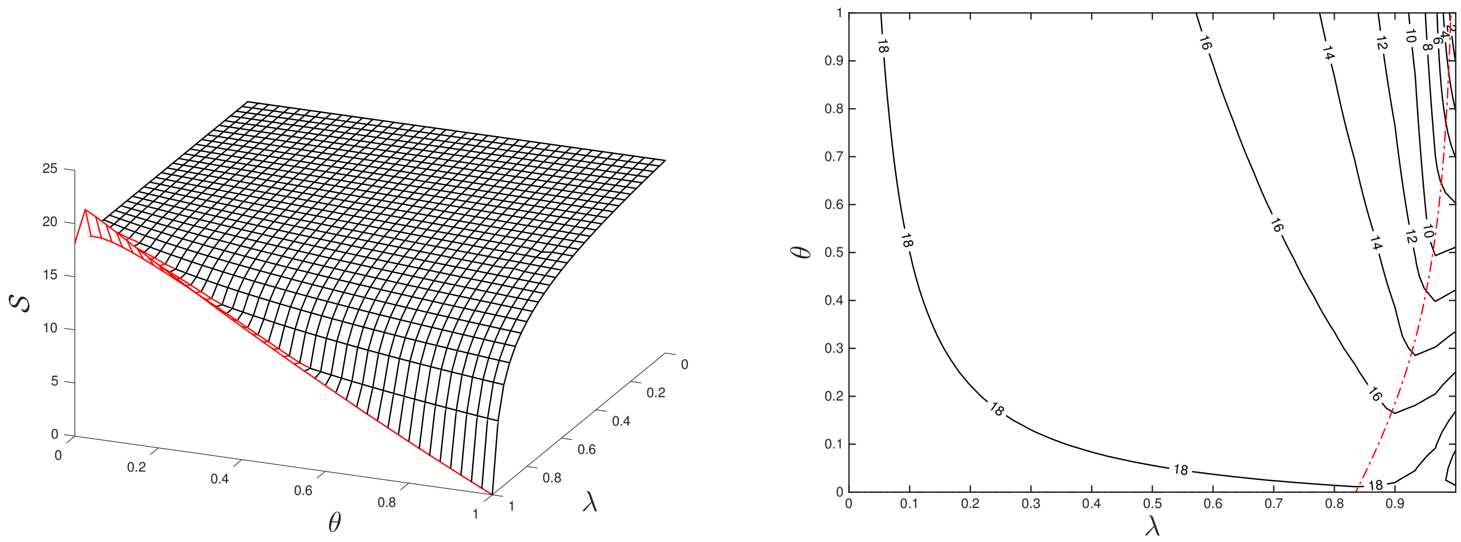
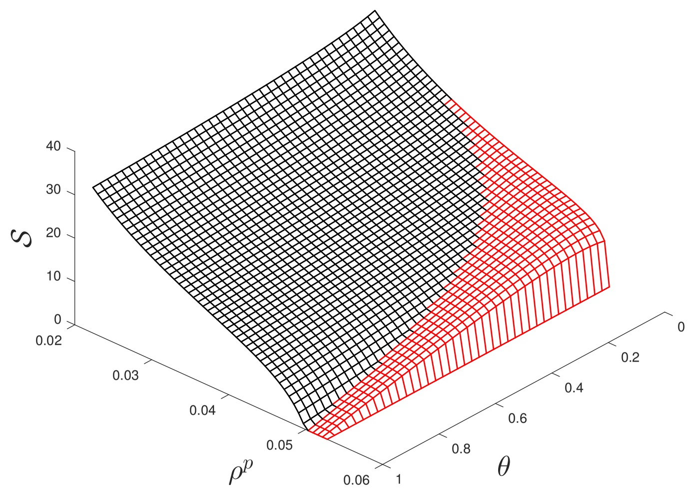


Figure 28: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).



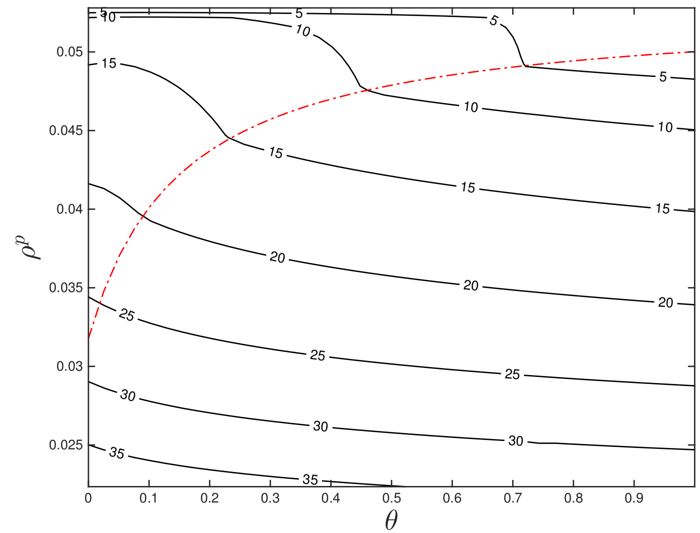


Figure 29: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).