Technical Appendix for
Local Currency Pricing and International Tax Policies

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1 Model and Linearization

1.1 LCP model:

The model follows Engel (2011). Detailed structure of home country is described. Foreign country has an identical structure. Where appropriate, foreign variables are indicated with asterisk.

1.1.1 Demand:

The household's preference is given by:

$$U(h) = E_{t=0}^{\infty}\beta^t(\frac{C_t^{1-\rho}}{1-\rho} - \eta L_t) \tag{1.1}$$

$$C_t = [C_{h,t}]^{\frac{v}{2}}[C_{f,t}]^{1-\frac{v}{2}} \tag{1.2}$$

The budget constraint:

$$P_tC_t + B_{ht+1} + \sum_{\zeta^{t+1}\in Z_{t+1}} B(\zeta^{t+1}|\zeta^t)D(\zeta^{t+1}) = W_tL_t + R_{t-1}B_{ht} + \Pi_t + D(\zeta^t) \tag{1.3}$$

The first order conditions for bond holding and labor supply

$$\frac{1}{R_t} = \beta E_t \frac{C_t^{\rho}P_t}{C_{t+1}^{\rho}P_{t+1}} \tag{1.4}$$

$$W_t\frac{C_t^{-\rho}}{P_t} = \eta \tag{1.5}$$

Under complete market assumption, we have

$$\frac{C_t^{-\rho}}{P_t} = \Gamma\frac{(C_t^*)^{-\rho}}{S_tP_t^*} \tag{1.6}$$

In a symmetrical model, $\Gamma = 1$. The demand of home and foreign good

$$C_{h,t} = \frac{v}{2}\frac{P_tC_t}{P_{hht}} \tag{1.7}$$

$$C_{f,t} = (1-\frac{v}{2})\frac{P_tC_t}{P_{fht}(1+t_{c,t})} \tag{1.8}$$

where $t_{c,t}$ is state-contingent consumption tax on imported goods.

1.1.2 Supply:

Each differentiated good firm in the home country has the following production function:

$$y_t = Z_t L_t \tag{1.9}$$

This yileds marginal cost:

$$mc_t = \frac{MC_t}{P_t} = \frac{W_t}{P_t Z_t} \tag{1.10}$$

The optimization problem for each differentiated good film i can be described below,

$$E_t \sum_{j=0}^{\infty} \kappa^j \beta_{t,t+j}(1+\tau)[(P^o_{hht}(i) - MC_{t+j}(i))C_{ht+j}(i) + (S_{t+j}(1+s_{e,t+j})P^{o*}_{hft}(i) - MC_{t+j}(i))C^*_{ht+j}(i)]$$

where $s_{e,t}$ is state-contingent export subsidy. τ is tax (subsidy) to solve monopolistic distortion and satisfy $1+\tau = \frac{\lambda}{\lambda-1}$. demand of varieties:

$$C^*_{ht+j}(i) = (\frac{P^{o*}_{hft}(i)}{P^*_{hft+j}})^{-\lambda} C^*_{ht+j} \; ; C_{ht+j}(i) = (\frac{P^o_{hht}(i)}{P_{hht+j}})^{-\lambda} C_{ht+j}$$

For the convenience of analysis, we define the real price $p_x = \frac{P_x}{P_t}$ and the real exchange rate $e_{t+j} = \frac{S_{t+j}P^*_{t+j}}{P_{t+j}}$

The optimal pricing:

$$\begin{aligned}
P^o_{hht}(i) &= \frac{1}{1+\tau}\frac{\lambda}{\lambda-1}\frac{E_t \sum_{j=0}^{\infty} \kappa^j \beta_{t,t+j} P^{\lambda}_{hht+j} MC_{t+j} C_{ht+j}}{E_t \sum_{j=0}^{\infty} \kappa^j \beta_{t,t+j} P^{\lambda}_{hht+j} C_{ht+j}} \\
P^{o*}_{hft}(i) &= \frac{1}{1+\tau}\frac{\lambda}{\lambda-1}\frac{E_t \sum_{j=0}^{\infty} \kappa^j \beta_{t,t+j} (P^*_{hft+j})^{\lambda} MC_{t+j} C^*_{ht+j}}{E_t \sum_{j=0}^{\infty} \kappa^j S_{t+j}(1+s_{e,t+j})\beta_{t,t+j} (P^*_{hft+j})^{\lambda} C^*_{ht+j}}.
\end{aligned}$$

we detrend the price by average price level:

$$\begin{aligned}
\frac{P^o_{hht}(i)}{P_{hht}} &= \frac{1}{1+\tau}\frac{\lambda}{\lambda-1}\frac{E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C^{-\rho}_{t+j}}{C^{-\rho}_t} (\frac{P_{hht+j}}{P_{hht}})^{\lambda} mc_{t+j} C_{ht+j}}{E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C^{-\rho}_{t+j}}{C^{-\rho}_t} (\frac{P_{hht+j}}{P_{hht}})^{\lambda-1} C_{ht+j} p_{hht+j}} = \frac{K_{hh,t}}{F_{hh,t}} \\
\frac{P^{o*}_{hft}(i)}{P^*_{hft}} &= \frac{1}{1+\tau}\frac{\lambda}{\lambda-1}\frac{E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C^{-\rho}_{t+j}}{C^{-\rho}_t} (\frac{P^*_{hft+j}}{P^*_{hft}})^{\lambda} mc_{t+j} C^*_{ht+j}}{E_t \sum_{j=0}^{\infty} (\beta\kappa)^j e_{t+j}(1+s_{e,t+j}) \frac{C^{-\rho}_{t+j}}{C^{-\rho}_t} (\frac{P^*_{hft+j}}{P^*_{hft}})^{\lambda-1} C^*_{ht+j} p^*_{hft+j}} = \frac{K^*_{hf,t}}{F^*_{hf,t}}
\end{aligned}$$

$$K_{hh,t} = E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C_{t+j}^{-\rho}}{C_t^{-\rho}} (\frac{P_{hht+j}}{P_{hht}})^{\lambda} mc_{t+j} C_{ht+j} = C_{h,t} mc_t + \beta\kappa E_t \frac{C_{t+1}^{-\rho}}{C_t^{-\rho}} \pi_{hh,t}^{\lambda} K_{hh,t+1} \tag{1.11}$$

$$F_{hh,t} = E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C_{t+j}^{-\rho}}{C_t^{-\rho}} (\frac{P_{hht+j}}{P_{hht}})^{\lambda-1} C_{ht+j} p_{hht+j} = C_{h,t} p_{hh,t} + \beta\kappa E_t \frac{C_{t+1}^{-\rho}}{C_t^{-\rho}} \pi_{hh,t}^{\lambda-1} F_{hh,t+1} \tag{1.12}$$

$$\begin{aligned} K_{hf,t}^{*} &= E_t \sum_{j=0}^{\infty} (\beta\kappa)^j \frac{C_{t+j}^{-\rho}}{C_t^{-\rho}} (\frac{P_{hft+j}^{*}}{P_{hft}^{*}})^{\lambda} mc_{t+j} C_{ht+j}^{*} && (1.13) \\ &= C_{h,t}^{*} mc_t + \beta\kappa E_t \frac{C_{t+1}^{-\rho}}{C_t^{-\rho}} (\pi_{hf,t}^{*})^{\lambda} K_{hf,t+1}^{*} \\ F_{hf,t}^{*} &= E_t \sum_{j=0}^{\infty} (\beta\kappa)^j e_{t+j} (1 + s_{e,t+j}) \frac{C_{t+j}^{-\rho}}{C_t^{-\rho}} (\frac{P_{hft+j}^{*}}{P_{hft}^{*}})^{\lambda-1} C_{ht+j}^{*} p_{hft+j}^{*} && (1.14) \\ &= C_{ht}^{*} p_{hft}^{*} e_t (1 + s_{e,t}) + \beta\kappa E_t \frac{C_{t+1}^{-\rho}}{C_t^{-\rho}} (\pi_{hf,t}^{*})^{\lambda-1} F_{hf,t+1}^{*} \end{aligned}$$

Follow Fujiwara and Wang (2016), we specify price dispersion term $\Delta_{hh,t}$ and $\Delta_{hf,t}^{*}$ so as to solve second order approximated loss function.

$$\Delta_{hh,t} = \int_0^1 (\frac{P_{hht}(i)}{P_{hht}})^{-\lambda} di = (1-\kappa)(\frac{P_{hht}^{o}}{P_{hht}})^{-\lambda} + \kappa \pi_{hh,t}^{\lambda} \Delta_{hh,t-1} \tag{1.15}$$

$$\Delta_{hf,t}^{*} = \int_0^1 (\frac{P_{hft}^{*}(i)}{P_{hft}^{*}})^{-\lambda} di = (1-\kappa)(\frac{P_{hft}^{*o}}{P_{hft}^{*}})^{-\lambda} + \kappa (\pi_{hf,t}^{*})^{\lambda} \Delta_{hf,t-1}^{*} \tag{1.16}$$

Since a fraction κ of goods prices remain unchanged from the previous period, the price index for home goods sold in the home and foreign markets can be written as follows, respectively,

$$\begin{aligned} P_{hht} &= [\kappa P_{hht-1}^{1-\lambda} + (1-\kappa)(P_{hht}^{o})^{1-\lambda}]^{\frac{1}{1-\lambda}}, \\ &\implies 1 = [\kappa \pi_{hh,t}^{\lambda-1} + (1-\kappa)(\frac{P_{hht}^{o}}{P_{hht}})^{1-\lambda}]^{\frac{1}{1-\lambda}} && (1.17) \end{aligned}$$

$$\begin{aligned} P_{hft}^{*} &= [\kappa P_{hft-1}^{*1-\lambda} + (1-\kappa)(P_{hft}^{*o})^{1-\lambda}]^{\frac{1}{1-\lambda}}, \\ &\implies 1 = [\kappa (\pi_{hf,t}^{*})^{\lambda-1} + (1-\kappa)(\frac{P_{hht}^{*o}}{P_{hht}^{*}})^{1-\lambda}]^{\frac{1}{1-\lambda}} && (1.18) \end{aligned}$$

Given these prices and consumption tax rates, price index is:

$$1 = (\frac{v}{2})^{\frac{v}{2}}(1-\frac{v}{2})^{1-\frac{v}{2}}(\frac{P_{hht}}{P_t})^{\frac{v}{2}}((1+t_{c,t})\frac{P_{fht}}{P_t})^{1-\frac{v}{2}} \tag{1.19}$$

We also define the inflation for domestically produced goods and imported goods, respectively, $\pi_{hh,t} = \frac{P_{hh,t}}{P_{hh,t-1}}, \pi^*_{hf,t} = \frac{P^*_{hf,t}}{P^*_{hf,t-1}}$.

$$\pi_{hh,t} = \pi_t \frac{p_{hht}}{p_{hht-1}} \tag{1.20}$$

$$\pi^*_{hft} = \pi^*_t \frac{p^*_{hft}}{p^*_{hft-1}} \tag{1.21}$$

where real price $p_{hht} = \frac{P_{hht}}{P_t}$.

1.1.3 Equilibrium

In equilibrium, goods market clearing condition implies that

$$Y_t = \frac{v}{2}\frac{P_t C_t}{P_{hht}}\Delta_{hh,t} + (1-\frac{v}{2})\frac{P^*_t C^*_t}{P^*_{hft}(1+t^*_{c,t})}\Delta^*_{hf,t} \tag{1.22}$$

Equations (1.4)-(1.22), together with the similar equations in foreign economy, monetary policy rules, consumption tax policy and export subsidy policy, and the risk sharing condition (1.6), define the two country economy.

1.2 First order log-linearization

Monopolistic distortion is corrected by production tax $1+\tau = \frac{\lambda}{\lambda-1}$, so steady state in our model is efficient. We follow Fujiwara and Wang (2016) to derive our loss function. Most of equations are same as those in their paper, the appendix is to highlight derivation when tax and subsidy are introduced. In our notation, $\widehat{X_t} = \log(X_t) - \log(\overline{X})$ indicates the log-deviation of a variable (X_t) from the respective steady state ($\overline{X}$). The first order log-linearization of equations are present below.

- the deviation from law of one price

$$\widehat{d_t} = \widehat{p}^*_{hf,t} + \widehat{e}_t - \widehat{p}_{hh,t} \tag{1.23}$$

$$\widehat{d}^*_t = \widehat{p}_{fh,t} - \widehat{e}_t - \widehat{p}^*_{ff,t} \tag{1.24}$$

- term of trade

$$\widehat{q}_t = \widehat{p}_{fh,t} - \widehat{p}^*_{hf,t} - \widehat{e}_t \tag{1.25}$$

$$\widehat{q}^*_t = -\widehat{q}_t \tag{1.26}$$

- log-linear form of price index:

$$\frac{v}{2}\widehat{p}_{hh,t} + \frac{2-v}{2}(\widehat{p}_{fh,t} + \widehat{t}_{c,t}) = 0 \tag{1.27}$$

$$\frac{v}{2}\widehat{p}^*_{ff,t} + \frac{2-v}{2}(\widehat{p}^*_{hf,t} + \widehat{t}^*_{c,t}) = 0 \tag{1.28}$$

where $\widehat{t}_{c,t} = \log(1 + t_{c,t})$ and $\widehat{t}^*_{c,t} = \log(1 + t^*_{c,t})$ because $\bar{t}_c = \bar{t}^*_c = 0$.

- inflation dynamics

$$\pi_{hh,t} = \pi_t + \widehat{p}_{hh,t} - \widehat{p}_{hh,t-1} \tag{1.29}$$

$$\pi^*_{hf,t} = \pi^*_t + \widehat{p}^*_{hf,t} - \widehat{p}^*_{hf,t-1} \tag{1.30}$$

$$\pi^*_{ff,t} = \pi^*_t + \widehat{p}^*_{ff,t} - \widehat{p}^*_{ff,t-1} \tag{1.31}$$

$$\pi_{fh,t} = \pi_t + \widehat{p}_{fh,t} - \widehat{p}_{fh,t-1} \tag{1.32}$$

- optimal choices

$$\rho E_t\widehat{C}_{t+1} = \rho\widehat{C}_t + \widehat{R}_t - E_t\pi_{t+1} \tag{1.33}$$

$$\widehat{w}_t = \rho\widehat{C}_t \tag{1.34}$$

- demand of home and foreign good

$$\widehat{C}_{h,t} = -\widehat{p}_{hh,t} + \widehat{C}_t \tag{1.35}$$

$$\widehat{C}_{f,t} = -\widehat{p}_{fh,t} + \widehat{C}_t - \widehat{t}_{c,t} \tag{1.36}$$

$$\widehat{C}^*_{h,t} = -\widehat{p}^*_{hf,t} + \widehat{C}^*_t \tag{1.37}$$

$$\widehat{C}^*_{h,t} = -\widehat{p}^*_{ff,t} + \widehat{C}^*_t - \widehat{t}_{c,t} \tag{1.38}$$

- production function and marginal cost

$$\widehat{mc}_t = \widehat{w}_t - \theta_t \tag{1.39}$$

$$\widehat{mc}^*_t = \widehat{w}^*_t - \theta^*_t \tag{1.40}$$

$$\widehat{Y}_t = \theta_t + \widehat{L}_t \tag{1.41}$$

$$\widehat{Y}^*_t = \theta^*_t + \widehat{L}_t \tag{1.42}$$

- Market clearing condition:

$$\widehat{Y}_t = \widehat{C}_t - \frac{v}{2}\widehat{p}_{hh,t} - \frac{2-v}{2}(\widehat{p}^*_{hf,t} + \widehat{t}^*_{c,t}) - \frac{2-v}{2}\frac{1}{\rho}\widehat{e}_t \tag{1.43}$$

$$\widehat{Y}^*_t = \widehat{C}^*_t - \frac{v}{2}\widehat{p}^*_{ff,t} - \frac{2-v}{2}(\widehat{p}_{fh,t} + \widehat{t}_{c,t}) + \frac{2-v}{2}\frac{1}{\rho}\widehat{e}_t \tag{1.44}$$

$$\widehat{Y}_t - \widehat{Y}^*_t = -\frac{v}{2}\widehat{p}_{hh,t} - \frac{2-v}{2}(\widehat{p}^*_{hf,t}) + \frac{v-1}{\rho}\widehat{e}_t + \frac{v}{2}\widehat{p}^*_{ff,t} + \frac{2-v}{2}\widehat{p}_{fh,t} + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \tag{1.45}$$

- marginal cost in term of producer price index:

$$\begin{aligned}\widehat{mc}_t - \widehat{p}_{hh,t} &= \widehat{w}_t - \theta_t - \widehat{p}_{hh,t} \\ &= \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \qquad (1.46) \\ &\quad + \frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t})\end{aligned}$$

Note that from fourth equation to fifth equation, we use log-linear form of price index (1.27) and (1.28). From fifth equation to sixth equation, we use defnition of $\widehat{q}_t$ and $\widehat{d}_t$. Comparing with equation (6.143) in Fujiwara and Wang (2016), we can find that real marginal cost (in term of producer price index) are affected by consumption tax. The intuition is straight forward: Tax will distort the consumption choice and has impact on wage, and is eventually reflected in marginal cost.

$$\begin{aligned}&\widehat{mc}^*_t - \widehat{p}^*_{ff,t} \\ = \; &\rho\widehat{Y}^*_t - \theta^*_t + \frac{2-v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) + \frac{(2-v)(1-\rho)}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})\end{aligned} \tag{1.47}$$

$$\begin{aligned}&\widehat{mc}^*_t - \widehat{p}_{fh,t} + \widehat{e}_t \\ = \; &\rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) + \frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t}\end{aligned} \tag{1.48}$$

$$\begin{aligned}&\widehat{mc}_t - \widehat{p}^*_{hf,t} - \widehat{e}_t \\ = \; &\rho\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) - \frac{(2-v)}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}^*_{c,t}\end{aligned} \tag{1.49}$$

- risk sharing condition

$$\widehat{C}_t - \widehat{C}_t^* = \frac{1}{\rho}\widehat{e}_t \tag{1.50}$$

1.3 Second-order approximations

1.3.1 Price dispersion terms

price dispersion terms are same as Fujiwara and Wang (2016).

$$\begin{aligned}
\Delta_{hh,t} &= \int_0^1 (\frac{P_{hh,t}(j)}{P_{hh,t}})^{-\lambda} di \\
E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\widehat{\Delta}_{hh,t} &= E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\frac{\lambda}{2\delta}\pi_{hh,t}^2 \qquad (1.51) \\
\Delta_{hf,t}^* &= \int_0^1 (\frac{P_{hf,t}^*(j)}{P_{hf,t}^*})^{-\lambda} di \\
E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\widehat{\Delta}_{hf,t}^* &= E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\frac{\lambda}{2\delta}(\pi_{hf,t}^*)^2 \qquad (1.52) \\
\Delta_{ff,t}^* &= \int_0^1 (\frac{P_{ff,t}^*(j)}{P_{ff,t}^*})^{-\lambda} di \\
E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\widehat{\Delta}_{ff,t}^* &= E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\frac{\lambda}{2\delta}(\pi_{ff,t}^*)^2 \qquad (1.53) \\
\Delta_{fh,t} &= \int_0^1 (\frac{P_{fh,t}(j)}{P_{fh,t}})^{-\lambda} di \\
E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\widehat{\Delta}_{fh,t} &= E_{t_0}\sum_{t=t_0}^{\infty}(\beta\kappa)^{t-t_0}\frac{\lambda}{2\delta}(\pi_{fh,t})^2 \qquad (1.54)
\end{aligned}$$

where $\delta = \frac{(1-\beta\kappa)(1-\kappa)}{\kappa}$. Please refer to Fujiwara and Wang (2016) for the technical details.

1.3.2 AS equation

A standard method to derive second order approximated AS equation is documented in Appendix B.4 in Benigno and Woodford (2005, hereafter BW05). Here we highlight the key deviation when subsidies are introduced. Second order approximation of (1.11) and

(1.12):

$$\widehat{K}_{hh,t} + \frac{1}{2}\widehat{K}^2_{hh,t} = (1-\beta\kappa)E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{k}_{hht,T} + \frac{1}{2}\widehat{k}^2_{hht,T}) \tag{1.55}$$

$$\widehat{k}_{hht,T} = \widehat{k}_{hh,T} + \lambda\sum_{s=t+1}^{T}\pi_{hh,s} \tag{1.56}$$

$$\widehat{k}_{hh,T} = -\rho\widehat{C}_T + \widehat{C}_{h,T} + \widehat{mc}_T \tag{1.57}$$

$$\widehat{F}_{hh,t} + \frac{1}{2}\widehat{F}^2_{hh,t} = (1-\beta\kappa)E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{f}_{hht,T} + \frac{1}{2}\widehat{f}^2_{hht,T}) \tag{1.58}$$

$$\widehat{f}_{hht,T} = \widehat{f}_{hh,T} + (\lambda-1)\sum_{s=t+1}^{T}\pi_{hh,s} \tag{1.59}$$

$$\widehat{f}_{hh,T} = -\rho\widehat{C}_T + \widehat{C}_{h,T} + \widehat{p}_{hh,T} \tag{1.60}$$

$$\widehat{k}_{hh,T} - \widehat{f}_{hh,T} = \widehat{mc}_T - \widehat{p}_{hh,T} \tag{1.61}$$

$$\widehat{k}_{hh,T} + \widehat{f}_{hh,T} = \widehat{mc}_T + \widehat{p}_{hh,T} + 2(-\rho\widehat{C}_T + \widehat{C}_{h,T}) \tag{1.62}$$

$$\widehat{k}^2_{hh,T} - \widehat{f}^2_{hh,T} = (\widehat{mc}_T - \widehat{p}_{hh,T})(\widehat{mc}_T + \widehat{p}_{hh,T} + 2(-\rho\widehat{C}_T + \widehat{C}_{h,T})) \tag{1.63}$$

$\widehat{k}_{hht,T}, \widehat{k}_{hh,T}, \widehat{f}_{hht,T}, \widehat{f}_{hh,T}$ are introduced to keep our notation consistent with BW05. The optimal pricing rule for home producer at domestic market:

$$1 = [\kappa\pi_{hh,t}^{\lambda-1} + (1-\kappa)(\frac{K_{hh,t}}{F_{hh,t}})^{1-\lambda}]^{\frac{1}{1-\lambda}}$$

Following Appendix B.4 and mainly equation (B.44) in BW05, we have AS equation:

$$t.i.p = \delta E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{k}_{hh,T} - \widehat{f}_{hh,T} + \frac{1}{2}\widehat{k}^2_{hh,T} - \frac{1}{2}\widehat{f}^2_{hh,T}) + \frac{1}{2}\lambda E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}\pi^2_{hh,T}$$

By subsituting expression of $\widehat{k}_{hh,T} - \widehat{f}_{hh,T}$ and $\frac{1}{2}\widehat{k}^2_{hh,T} - \frac{1}{2}\widehat{f}^2_{hh,T}$

$$E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{mc}_T - \widehat{p}_{hh,T}) \tag{1.64}$$

$$= -E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}\left[\frac{1}{2}(\widehat{mc}_T - \widehat{p}_{hh,T})(\widehat{mc}_T + \widehat{p}_{hh,T} + 2(-\rho\widehat{C}_T + \widehat{C}_{h,T})) + \frac{\lambda}{2\delta}\pi^2_{hh,T}\right] + t.i.p$$

$t.i.p$ is term independent of policy. $\delta = \frac{(1-\beta\kappa)(1-\kappa)}{\kappa}$. Similarly

$$\widehat{K}^{*}_{hf,t} + \frac{1}{2}\widehat{K}^{*2}_{hf,t} = (1-\beta\kappa)E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{k}^{*}_{hft,T} + \frac{1}{2}\widehat{k}^{*2}_{hft,T}) \tag{1.65}$$

$$\widehat{k}^{*}_{hft,T} = \widehat{k}^{*}_{hf,T} + \lambda\sum_{s=t+1}^{T}\pi^{*}_{hf,s} \tag{1.66}$$

$$\widehat{k}^{*}_{hfT} = -\rho\widehat{C}_T + \widehat{C}^{*}_{h,T} + \widehat{mc}_T \tag{1.67}$$

$$\widehat{F}^{*}_{hf,t} + \frac{1}{2}\widehat{F}^{*2}_{hf,t} = (1-\beta\kappa)E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{f}^{*}_{hft,T} + \frac{1}{2}\widehat{f}^{*2}_{hft,T}) \tag{1.68}$$

$$\widehat{f}^{*}_{hft,T} = \widehat{f}^{*}_{hf,T} + (\lambda-1)\sum_{s=t+1}^{T}\pi^{*}_{hf,s} \tag{1.69}$$

$$\widehat{f}_{hf,T} = -\rho\widehat{C}_T + \widehat{C}^{*}_{h,T} + \widehat{p}^{*}_{hf,T} + \widehat{e}_T + \widehat{s}_{e,T} \tag{1.70}$$

$$\widehat{k}^{*}_{hf,T} - \widehat{f}^{*}_{hf,T} = \widehat{mc}_T - \widehat{p}^{*}_{hf,T} - \widehat{e}_T - \widehat{s}_{e,T} \tag{1.71}$$

$$\widehat{k}^{*}_{hf,T} + \widehat{f}^{*}_{hf,T} = \widehat{mc}_T + \widehat{p}^{*}_{hf,T} + 2(-\rho\widehat{C}_T + \widehat{C}^{*}_{h,T}) + \widehat{e}_T + \widehat{s}_{e,T} \tag{1.72}$$

$$\widehat{k}^{*2}_{hf,T} - \widehat{f}^{*2}_{hf,T} = (\widehat{mc}_T - \widehat{p}^{*}_{hf,T} - \widehat{e}_T - \widehat{s}_{e,T})(\widehat{mc}_T + \widehat{p}^{*}_{hf,T} + 2(-\rho\widehat{C}_T + \widehat{C}^{*}_{h,T}) + \widehat{e}_T + \widehat{s}_{e,T}) \tag{1.73}$$

$$\begin{aligned} & E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{mc}_T - \widehat{p}^{*}_{hf,T} - \widehat{e}_T - \widehat{s}_{e,T}) + t.i.p \qquad (1.74) \\ = & -E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}\left[\frac{1}{2}(\widehat{mc}_T - \widehat{p}^{*}_{hf,T} - \widehat{e}_T - \widehat{s}_{e,T})(\widehat{mc}_T + \widehat{p}^{*}_{hf,T} + 2(-\rho\widehat{C}_T + \widehat{C}^{*}_{h,T}) + \widehat{e}_T + \widehat{s}_{e,T}) + \frac{\lambda}{2\delta}\pi^{*2}_{H,T}\right] \end{aligned}$$

Comparing with (6.96) in Fujiwara and Wang (2016), we find that subsidy enters into the aggregate supply decision to export. For foreign country, the AS equations to supply domestic market and export market are

$$\begin{aligned} & E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}(\widehat{mc}^{*}_T - \widehat{p}^{*}_{ff,T}) + t.i.p \qquad (1.75) \\ = & -E_t\sum_{T=t}^{\infty}(\beta\kappa)^{T-t}\left[\frac{1}{2}(\widehat{mc}^{*}_T - \widehat{p}^{*}_{ff,T})(\widehat{mc}^{*}_T + \widehat{p}^{*}_{ff,T} + 2(-\rho\widehat{C}^{*}_T + \widehat{C}^{*}_{f,T})) + \frac{\lambda}{2\delta}\pi^{*2}_{ff,T}\right] \end{aligned}$$

$$
\begin{aligned}
& E_t \sum_{T=t}^{\infty} (\beta\kappa)^{T-t} (\widehat{mc}^*_T - \widehat{p}_{fh,T} + \widehat{e}_T - \widehat{s}^*_{e,T}) + t.i.p \qquad (1.76) \\
= & -E_t \sum_{T=t}^{\infty} (\beta\kappa)^{T-t} \left[\frac{1}{2} (\widehat{mc}^*_T - \widehat{p}_{fh,T} + \widehat{e}_T - \widehat{s}^*_{e,T})(\widehat{mc}^*_T - \widehat{p}_{fh,T} + \widehat{e}_T - \widehat{s}^*_{e,T} + 2(-\rho \widehat{C}^*_T + \widehat{C}_{f,T})) + \frac{\lambda}{2\delta} \pi^2_{fh,T} \right]
\end{aligned}
$$

To further simplify AS equation, we utilize first order conditions, namely equation (1.35)-(1.48) and replace real marginal cost in the preceding second order approximated AS equations. For AS equation (1.64), we use $\widehat{mc}_t - \widehat{p}_{hh,t} = \rho \widehat{C}_t - \widehat{p}_{hh,t} - \theta_t$. The simplification detail is following:

$$
\begin{aligned}
& \widehat{mc}_t - \widehat{p}_{hh,t} \\
= & -\frac{1}{2} (\widehat{mc}_T - \widehat{p}_{hh,t})(\widehat{mc}_T + \widehat{p}_{hh,t} + 2(-\rho \widehat{C}_T + \widehat{C}_{h,T})) - \frac{\lambda}{2\delta} \pi^2_{hh,t} \\
= & \frac{1}{2} (1-\rho)^2 \widehat{C}^2_T - \frac{1}{2} (\widehat{C}_t - \widehat{p}_{hh,t} - \theta_t)^2 - \frac{\lambda}{2\delta} \pi^2_{hh,t} \\
= & -\frac{1}{2} ((\widehat{Y}_t - \theta_t) + \frac{2-v}{2} (\widehat{p}^*_{hf,t} + \widehat{t}^{*c}_t - \widehat{p}_{hh,t} + \frac{1}{\rho} \widehat{e}_t))^2 + \frac{1}{2} (1-\rho)^2 \widehat{C}^2_t - \frac{\lambda}{2\delta} \pi^2_{hh,t} \qquad (1.77)
\end{aligned}
$$

Comparing with equation (6.110) in Fujiwara and Wang, we find that $\widehat{t}^{*c}_t$ enter into the first term. Potential this instrument helps to deal with impact of $\widehat{d}_t$ on marginal cost. For AS equation (1.74), we use $\widehat{mc}_t - \widehat{p}^*_{hf,t} - \widehat{e}_t - \widehat{s}_{e,t} = \rho \widehat{C}^*_t + \frac{v}{2-v} \widehat{p}^*_{ff,t} + \widehat{t}^{*c}_t - \widehat{s}_{e,t} - \theta_t$. The simplification detail is following:

$$
\begin{aligned}
& \widehat{mc}_t - \widehat{p}^*_{hf,t} - \widehat{e}_t - \widehat{s}_{e,t} \\
= & -\frac{1}{2} (\rho \widehat{C}_t - \widehat{p}^*_{hf,t} - \widehat{s}_{e,t} - \theta_t - \widehat{e}_t)(\rho \widehat{C}_t + \widehat{p}^*_{hf,t} + 2(-\rho \widehat{C}_t + \widehat{C}^*_{h,t}) + \widehat{e}_t + \widehat{s}_{e,t} - \theta_t) - \frac{\lambda}{2\delta} \pi^{*2}_{hf,t} \\
= & \frac{1}{2} \left[(1-\rho) \widehat{C}^*_t + \widehat{s}_{e,t} - \widehat{t}^*_{c,t} \right]^2 - \frac{\lambda}{2\delta} \pi^{*2}_{hf,t} - \frac{1}{2} \left[(\widehat{Y}_t - \theta_t) + \frac{v}{2} (\widehat{p}_{hh,t} - \widehat{p}^*_{hf,t} - \widehat{t}^*_{c,t} - \frac{1}{\rho} \widehat{e}_t) \right]^2 \qquad (1.78)
\end{aligned}
$$

Comparing with equation (6.111) in Fujiwara and Wang, we find that $\widehat{s}_{e,t} - \widehat{t}^*_{c,t}$ enter into the first term, and $\widehat{t}^*_{c,t}$ enter into third term. Similarly, we have

$$
\begin{aligned}
& \widehat{mc}^*_t - \widehat{p}^*_{ff,t} = \rho \widehat{C}^*_t - \widehat{p}^*_{ff,t} - \theta^*_t \\
= & -\frac{1}{2} ((\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2} (\widehat{p}_{fft} + \widehat{t}_{c,t} - \widehat{p}^*_{ff,t} - \frac{1}{\rho} \widehat{e}_t))^2 + \frac{1}{2} (1-\rho)^2 \widehat{C}^{*2}_t - \frac{\lambda}{2\delta} \pi^{*2}_{ff,t} \qquad (1.79)
\end{aligned}
$$

And

$$\widehat{mc}_t^* - \widehat{p}_{fh,t} + \widehat{e}_t - \widehat{s}_{e,t}^* = \rho\widehat{C}_t + \frac{v}{2-v}\widehat{p}_{hh,t} + \widehat{t}_{c,t} - \widehat{s}_{e,t}^* - \theta_t^*$$
$$= \frac{1}{2}\left[(1-\rho)\widehat{C}_t + \widehat{s}_{e,t}^* - \widehat{t}_{c,t}\right]^2 - \frac{\lambda}{2\delta}\pi_{fh,t}^2 - \frac{1}{2}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{p}_{ff,t} - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t)\right]^2 \tag{1.80}$$

1.3.3 Resource constraint

Market clearing condition:

$$\theta_t + \widehat{L}_t - \frac{v}{2}\widehat{C}_t - \frac{2-v}{2}\widehat{C}_t^* + \frac{v}{2}\widehat{p}_{hh,t} + \frac{2-v}{2}(\widehat{p}_{hf,t}^* + \widehat{t}_{c,t}^*)$$
$$= -\frac{1}{2}(\theta_t + \widehat{L}_t)^2 + \frac{v}{4}(\widehat{p}_{hh,t} - \widehat{C}_t)^2 + \frac{2-v}{4}(\widehat{p}_{hf,t}^* + \widehat{t}_{c,t}^* - \widehat{C}_t^*)^2 + \frac{v}{2}\widehat{\Delta}_{hh,t} + \frac{2-v}{2}\widehat{\Delta}_{hf,t}^* \tag{1.81}$$

up to first order condition:

$$\theta_t + \widehat{L}_t = \frac{v}{2}(\widehat{C}_t - \widehat{p}_{hh,t}) + \frac{2-v}{2}(\widehat{C}_t^* - \widehat{p}_{hf,t}^* - \widehat{t}_{c,t}^*))$$

And the square:

$$\begin{aligned}(\theta_t + \widehat{L}_t)^2 &= \frac{v^2}{4}(\widehat{C}_t - \widehat{p}_{hh,t})^2 + \frac{(2-v)v}{2}(\widehat{C}_t - \widehat{p}_{hh,t})(\widehat{C}_t^* - \widehat{p}_{hf,t}^* - \widehat{t}_{c,t}^*) \\ &\quad + \frac{(2-v)^2}{4}(\widehat{C}_t^* - \widehat{p}_{hf,t}^* - \widehat{t}_{c,t}^*)^2\end{aligned}$$

We replace $(\theta_t + \widehat{L}_t)^2$ in (1.81) and rearrange the term, then we have

$$E_{t_0}\sum_{t=t_0}^{\infty}(\beta)^{t-t_0}\left[\widehat{L}_t - \widehat{C}_t + \frac{2-v}{2}(\widehat{C}_t - \widehat{C}_t^*) + \frac{v}{2}(\widehat{p}_{hh,t} - \widehat{p}_{ff,t}^*)\right]$$
$$= E_{t_0}\sum_{t=t_0}^{\infty}(\beta)^{t-t_0}\left[\frac{v(2-v)}{8}(\widehat{p}_{hf,t}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)^2 + \frac{v\lambda}{4\delta}\pi_{hh,t}^2 + \frac{(2-v)\lambda}{4\delta}\pi_{hf,t}^{*2}\right] \tag{1.82}$$

Similarly,

$$E_{t_0}\sum_{t=t_0}^{\infty}(\beta)^{t-t_0}\left[\widehat{L}_t^* - \widehat{C}_t^* - \frac{2-v}{2}(\widehat{C}_t - \widehat{C}_t^*) - \frac{v}{2}(\widehat{p}_{hh,t} - \widehat{p}_{ff,t}^*)\right]$$
$$= E_{t_0}\sum_{t=t_0}^{\infty}(\beta)^{t-t_0}\left[\frac{v(2-v)}{8}(\widehat{p}_{fh,t} - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t)^2 + \frac{(2-v)\lambda}{4\delta}\pi_{fh,t}^2 + \frac{v\lambda}{4\delta}\pi_{ff,t}^{*2}\right] \tag{1.83}$$

Note that we drop θ_t because it's out of policy control.

1.4 Loss function:

$$
\begin{aligned}
W_{h,0} &= E_t \sum_{t=0}^{\infty} (\beta)^t \left[\frac{C_t^{1-\rho}}{1-\rho} - \eta L_t \right] \\
&\approx E_t \sum_{t=0}^{\infty} (\beta)^t \overline{C}^{1-\rho} \left[\widehat{C}_t - \widehat{L}_t + \frac{1-\rho}{2} \widehat{C}_t^2 - \frac{1}{2} \widehat{L}_t^2 \right]
\end{aligned} \tag{1.84}
$$

The key to solve loss function is to get $\widehat{C}_t - \widehat{L}_t$ and the method exactly follow Fujiwara ang Wang (2016). Our key equations are (1.77),(1.78),(1.79),(1.80),(1.82),(1.83). We define:

$$
\begin{aligned}
\rho \widehat{C}_t - \widehat{p}_{hh,t} &= f_1 \\
\rho \widehat{C}_t^* + \frac{v}{2-v} \widehat{p}_{ff,t}^* &= f_2 \\
\rho \widehat{C}_t^* - \widehat{p}_{ff,t}^* &= f_3 \\
\rho \widehat{C}_t + \frac{v}{2-v} \widehat{p}_{hh,t} &= f_4 \\
\widehat{L}_t - \widehat{C}_t + \frac{2-v}{2} (\widehat{C}_t - \widehat{C}_t^*) + \frac{v}{2} (\widehat{p}_{hh,t} - \widehat{p}_{ff,t}^*) &= f_5 \\
\widehat{L}_t^* - \widehat{C}_t^* - \frac{2-v}{2} (\widehat{C}_t - \widehat{C}_t^*) - \frac{v}{2} (\widehat{p}_{hh,t} - \widehat{p}_{ff,t}^*) &= f_6
\end{aligned}
$$

Namely, $f_1 - f_6$ are the RHS of (1.77)- (1.83). With several steps of calculation, we have

$$
\widehat{C}_t - \widehat{L}_t = -\frac{v}{2} \frac{2-v}{2} \frac{\rho - 1}{\rho} (f_1 - f_3) - \frac{v}{2} \frac{2-v}{2} \frac{\rho - 1 + \frac{2}{v}}{\rho} (f_2 - f_4) - f_5
$$

$$
\widehat{C}_t^* - \widehat{L}_t^* = \frac{v}{2} \frac{2-v}{2} \frac{\rho - 1}{\rho} (f_1 - f_3) + \frac{v}{2} \frac{2-v}{2} \frac{\rho - 1 + \frac{2}{v}}{\rho} (f_2 - f_4) - f_6
$$

Using (1.77) and (1.79), we have

$$
\begin{aligned}
f_1 - f_3 &= -\frac{1}{2} \left[(\widehat{Y}_t - \theta_t) + \frac{2-v}{2} (\widehat{p}_{hf,t}^* + \widehat{t}_t^{*c} - \widehat{p}_{hh,t} + \frac{1}{\rho} \widehat{e}_t) \right]^2 + \frac{1}{2} (1-\rho)^2 \widehat{C}_t^2 - \frac{\lambda}{2\delta} \pi_{hh,t}^2 \\
&\quad + \frac{1}{2} \left[(\widehat{Y}_t^* - \theta_t^*) + \frac{2-v}{2} (\widehat{p}_{fft} + \widehat{t}_{c,t} - \widehat{p}_{ff,t}^* - \frac{1}{\rho} \widehat{e}_t) \right]^2 - \frac{1}{2} (1-\rho)^2 \widehat{C}_t^{*2} + \frac{\lambda}{2\delta} \pi_{ff,t}^{*2}
\end{aligned}
$$

Using (1.78) and (1.80), we have

$$\begin{aligned} f_2 - f_4 &= \frac{1}{2}\left[(1-\rho)\widehat{C}_t^* + \widehat{s}_{e,t} - \widehat{t}_{c,t}^*\right]^2 - \frac{\lambda}{2\delta}\pi_{hf,t}^{*2} - \frac{1}{2}\left[(\widehat{Y}_t - \theta_t) + \frac{v}{2}(\widehat{p}_{hh,t} - \widehat{p}_{hf,t}^* - \widehat{t}_{c,t}^* - \frac{1}{\rho}\widehat{e}_t)\right]^2 \\ &\quad -\frac{1}{2}\left[(1-\rho)\widehat{C}_t + \widehat{s}_{e,t}^* - \widehat{t}_{c,t}\right]^2 + \frac{\lambda}{2\delta}\pi_{fh,t}^2 + \frac{1}{2}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{p}_{ff,t} - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t)\right]^2 \\ &\quad -\widehat{t}_t^{*c} + \widehat{s}_{e,t} + \widehat{t}_{c,t} - \widehat{s}_{e,t}^* \end{aligned}$$

$$f_5 = \frac{v(2-v)}{8}(\widehat{p}_{hf,t}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)^2 + \frac{v\lambda}{4\delta}\pi_{hh,t}^2 + \frac{(2-v)\lambda}{4\delta}\pi_{hf,t}^{*2}$$

$$f_6 = \frac{v(2-v)}{8}(\widehat{p}_{fh,t} - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t)^2 + \frac{(2-v)\lambda}{4\delta}\pi_{fh,t}^2 + \frac{v\lambda}{4\delta}\pi_{ff,t}^{*2}$$

Note that we drop θ_t because it's out of policy control. We have

$$\begin{aligned} -(\widehat{C}_t - \widehat{L}_t) &= \frac{v(2-v)}{8}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)^2 \\ &\quad + \frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{hh,t}^2 + \frac{(2-v)}{2}\pi_{fh,t}^2\right] \\ &\quad + \frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ &\quad - \frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right)^2 \\ &\quad + \frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t^* + \widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right)^2 \\ &\quad + \frac{v(2-v)}{8}\frac{\rho-1}{\rho}(1-\rho)^2(\widehat{C}_t^2 - \widehat{C}_t^{*2}) \\ &\quad + \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{t}_{c,t} - \widehat{s}_{e,t}^*) - (\widehat{t}_{c,t}^* - \widehat{s}_{e,t})\right) \\ &\quad + \frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2 \\ &\quad + \frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\ &\quad - \frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\ &\quad - \frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2 \end{aligned} \tag{1.85}$$

where $\alpha = 1 + (1-\rho)(1-v)$

And corresponding loss function is

$$
\begin{aligned}
L_{h,0} &= -W_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ -(\widehat{C}_t - \widehat{L}_t) - \frac{1-\rho}{2}\widehat{C}_t^2 + \frac{1}{2}\widehat{L}_t^2 \right\} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c}
\frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\
+\frac{v(2-v)}{8}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)^2 \\
+\frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{hh,t}^2 + \frac{(2-v)}{2}\pi_{fh,t}^2\right] \\
+\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\
-\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right)^2 - \frac{1-\rho}{2}\widehat{C}_t^2 \\
+\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t^* + \widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right)^2 \\
+\frac{v(2-v)}{8}\frac{\rho-1}{\rho}(1-\rho)^2(\widehat{C}_t^2 - \widehat{C}_t^{*2}) \\
+\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{t}_{c,t} - \widehat{s}_{e,t}^*) - (\widehat{t}_{c,t}^* - \widehat{s}_{e,t})\right) \\
+\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2 \\
+\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\
-\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\
-\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2
\end{array} \right\}
\end{aligned} \quad (1.86)
$$

foreign loss function is:

$$L_{h,0}^{*} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t^{*} - \theta_t^{*})^2 \\ +\frac{v(2-v)}{8}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^{*} + \widehat{t}_{c,t})^2 \\ +\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{hh,t}^2 + \frac{(2-v)}{2}\pi_{fh,t}^2\right] + \frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}_{e,t}^{*}\right)^2 - \frac{1-\rho}{2}\widehat{C}_t^{*2} \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t^{*} + \widehat{t}_{c,t}^{*} - \widehat{s}_{e,t}\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1}{\rho}(1-\rho)^2(\widehat{C}_t^2 - \widehat{C}_t^{*2}) \\ -\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{t}_{c,t} - \widehat{s}_{e,t}^{*}) - (\widehat{t}_{c,t}^{*} - \widehat{s}_{e,t})\right) \\ -\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}_{hft}^{*} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^{*})\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^{*} + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t^{*} - \theta_t^{*}) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^{*} + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}_{hft}^{*} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^{*})\right)^2 \end{array} \right\} \tag{1.87}$$

1.5 NKPC

Define:

$$\begin{aligned} Z_{hh,t} &= E_t \sum_{T=t}^{\infty} (\beta\kappa)^{T-t}\left[\widehat{k}_{hht,T} + \widehat{f}_{hht,T}\right] \\ &= E_t \sum_{T=t}^{\infty} (\beta\kappa)^{T-t}\left[\widehat{k}_{hh,T} + \widehat{f}_{hh,T} + (2\lambda - 1)\sum_{s=t+1}^{T}\pi_{hh,s}\right] \\ &= \widehat{k}_{hh,T} + \widehat{f}_{hh,T} + \beta\kappa E_t Z_{hh,t+1} \end{aligned}$$

where

$$\begin{aligned}
\widehat{k}_{hht,T} &= \widehat{k}_{hh,T} + \lambda \sum_{s=t+1}^{T} \pi_{hh,s} \\
\widehat{f}_{hht,T} &= \widehat{f}_{hh,T} + (\lambda - 1) \sum_{s=t+1}^{T} \pi_{hh,s} \\
\widehat{k}_{hh,T} &= -\rho \widehat{C}_T + \widehat{C}_{h,T} + \widehat{mc}_T \\
\widehat{f}_{hh,T} &= -\rho \widehat{C}_T + \widehat{C}_{h,T} + \widehat{p}_{hh,T} \\
\widehat{k}_{hh,T} + \widehat{f}_{hh,T} &= 2(-\rho \widehat{C}_T + \widehat{C}_{h,T}) + \widehat{mc}_T + \widehat{p}_{hh,T}
\end{aligned}$$

$$V_{hh,t} = \pi_{hh,t} - \frac{1}{2}\frac{1-\lambda}{1-\kappa}\pi^2_{hh,t} + \frac{1}{2}(1-\kappa\beta)\pi_{hh,t}Z_{hh,t} + \frac{\lambda}{2}\pi^2_{hh,t}$$

Follow Benigno and Woodford (2005), let $x_{hh,t} = \widehat{mc}_t - \widehat{p}_{hh,t}$. AS equation is

$$\begin{aligned}
V_{hh,t} &= \delta[\widehat{k}_{hh,t} - \widehat{f}_{hh,t} + \frac{1}{2}(\widehat{k}^2_{hh,t} - \widehat{f}^2_{hh,t}) + \frac{\lambda}{2\delta}\pi_{hh,t}] + \beta E_t V_{hh,t+1} \\
&= \delta x_{hh,t} + \frac{\delta}{2} x_{hh,t}(\widehat{mc}_t + \widehat{p}_{hh,t} + 2(-\rho \widehat{C}_t + \widehat{C}_{h,t})) + \frac{\lambda}{2}\pi^2_{hh,t} + \beta E_t V_{hh,t+1}
\end{aligned}$$

Up to first order approximation:

$$\pi_{hh,t} = V_{hh,t} = \delta(\widehat{mc}_t - \widehat{p}_{hh,t}) + \beta E_t \pi_{hh,t+1} \tag{1.88}$$

For other three NKPCs:

$$\begin{aligned}
\pi^*_{hf,t} &= \delta(\widehat{mc}_t - \widehat{p}^*_{hf,t} - \widehat{e}_t - \widehat{s}_{e,t}) + \beta E_t \pi^*_{hf,t+1} && (1.89) \\
\pi^*_{ff,t} &= \delta(\widehat{mc}^*_t - \widehat{p}^*_{ff,t}) + \beta E_t \pi^*_{ff,t+1} && (1.90) \\
\pi_{fh,t} &= \delta(\widehat{mc}^*_t - \widehat{p}_{fh,t} + \widehat{e}_t - \widehat{s}^*_{e,t}) + \beta E_t \pi_{fh,t+1} && (1.91)
\end{aligned}$$

Plugging the expression of marginal cost (1.46)-(1.49), we have

$$\pi_{hh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array} \right] + \beta E_t \pi_{hh,t+1} \quad (1.92)$$

$$\pi^*_{hf,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ -\frac{(2-v)}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}^*_{c,t} - \widehat{s}_{e,t} \end{array} \right] + \beta E_t \pi^*_{hf,t+1} \quad (1.93)$$

$$\pi^*_{ff,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}^*_t - \theta^*_t + \frac{2-v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}^*_{c,t} - \widehat{t}^c_t) \end{array} \right] + \beta E_t \pi^*_{ff,t+1} \quad (1.94)$$

$$\pi_{fh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ +\frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t} - \widehat{s}^*_{e,t} \end{array} \right] + \beta E_t \pi_{fh,t+1} \quad (1.95)$$

When $\rho = 1$, Phillips curve collapse into the corresponding NKPC curves in the paper.

2 Cooperation game $\rho = 1$

We consider the special case where $\rho = 1$. Home Loss function is, under noncoorperation assumption:

$$\begin{aligned}
L_{h,0} &= -W_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ -(\widehat{C}_t - \widehat{L}_t) - \frac{1-\rho}{2} \widehat{C}_t^2 + \frac{1}{2} \widehat{L}_t^2 \right\} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\ + \frac{v(2-v)}{8} (\widehat{d}_t + \widehat{t}^*_{c,t})^2 \\ + \frac{\lambda}{2\delta} \left[\frac{v}{2} \pi^2_{hh,t} + \frac{(2-v)}{2} \pi^2_{fh,t} \right] \\ -(\frac{2-v}{4}) \left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t} \right)^2 + (\frac{2-v}{4}) \left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t} \right)^2 \\ + \frac{(2-v)}{2} \left((\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) - (\widehat{t}^*_{c,t} - \widehat{s}_{e,t}) \right) - \theta_t \\ + \frac{2-v}{4} \left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2} (\widehat{d}^*_t + \widehat{t}_{c,t}) \right)^2 \\ - \frac{2-v}{4} \left((\widehat{Y}_t - \theta_t) - \frac{v}{2} (\widehat{d}_t + \widehat{t}^*_{c,t}) \right)^2 \end{array} \right\}
\end{aligned} \tag{2.96}$$

Similarly, foreign loss function is

$$\begin{aligned}
L^*_{h,0} &= -W^*_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ -(\widehat{C}^*_t - \widehat{L}^*_t) - \frac{1-\rho}{2} \widehat{C}^{*2}_t + \frac{1}{2} \widehat{L}^{*2}_t \right\} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 \\ + \frac{v(2-v)}{8} (\widehat{d}^*_t + \widehat{t}_{c,t})^2 \\ + \frac{\lambda}{2\delta} \left[\frac{v}{2} \pi^{*2}_{ff,t} + \frac{(2-v)}{2} \pi^{*2}_{hf,t} \right] \\ +(\frac{2-v}{4}) \left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t} \right)^2 - (\frac{2-v}{4}) \left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t} \right)^2 \\ - \frac{(2-v)}{2} \left((\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) - (\widehat{t}^*_{c,t} - \widehat{s}_{e,t}) \right) - \theta^*_t \\ - \frac{2-v}{4} \left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2} (\widehat{d}^*_t + \widehat{t}_{c,t}) \right)^2 \\ + \frac{2-v}{4} \left((\widehat{Y}_t - \theta_t) - \frac{v}{2} (\widehat{d}_t + \widehat{t}^*_{c,t}) \right)^2 \end{array} \right\}
\end{aligned} \tag{2.97}$$

The global loss function

$$\begin{aligned}
(P1) \quad &: \quad L^w = L_{h,0} + L^*_{h,0} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 + \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 - \theta_t - \theta^*_t \\ + \frac{v(2-v)}{8} (\widehat{d}^*_t + \widehat{t}_{c,t})^2 + \frac{v(2-v)}{8} (\widehat{d}_t + \widehat{t}^*_{c,t})^2 \\ + \frac{\lambda}{2\delta} \left[\frac{v}{2} \pi^{*2}_{ff,t} + \frac{(2-v)}{2} \pi^{*2}_{hf,t} \right] \\ + \frac{\lambda}{2\delta} \left[\frac{v}{2} \pi^2_{hh,t} + \frac{(2-v)}{2} \pi^2_{fh,t} \right] \end{array} \right\}
\end{aligned} \tag{2.98}$$

Note that $\left(\widehat{t}_{c,t}-\widehat{s}^{*}_{e,t}\right)^{2}$,$\left(\widehat{t}^{*}_{c,t}-\widehat{s}_{e,t}\right)^{2}$ and $(\widehat{t}_{c,t}-\widehat{s}^{*}_{e,t})-(\widehat{t}^{*}_{c,t}-\widehat{s}_{e,t})$ don't appear in loss function. The intuition is simple: under international coordination game, there are not costs of non-coordination. $\widehat{q}_{t}$ is term of trade effect. As in Clarida, Gali and Gertler (2002), term of trade effect does not appear in loss function in this special case ($\rho=1$). The central planner minimizes global loss function subject to (2.99)-(2.102) and (2.103).

$$\begin{aligned}
\pi_{hh,t} &= \delta\left[\widehat{Y}_{t}-\theta_{t}+\frac{2-v}{2}\widehat{d}_{t}+\frac{2-v}{2}\widehat{t}^{*}_{c,t}\right]+\beta E_{t}\pi_{hh,t+1} && (2.99)\\
\pi^{*}_{hf,t} &= \delta\left[\widehat{Y}_{t}-\theta_{t}-\frac{v}{2}\widehat{d}_{t}+\frac{2-v}{2}\widehat{t}^{*}_{c,t}-\widehat{s}_{e,t}\right]+\beta E_{t}\pi^{*}_{hf,t+1} && (2.100)\\
\pi^{*}_{ff,t} &= \delta\left[\widehat{Y}^{*}_{t}-\theta^{*}_{t}+\frac{2-v}{2}\widehat{d}^{*}_{t}+\frac{2-v}{2}\widehat{t}_{c,t}\right]+\beta E_{t}\pi^{*}_{ff,t+1} && (2.101)\\
\pi_{fh,t} &= \delta\left[\widehat{Y}^{*}_{t}-\theta^{*}_{t}-\frac{v}{2}\widehat{d}^{*}_{t}+\frac{2-v}{2}\widehat{t}_{c,t}-\widehat{s}^{*}_{e,t}\right]+\beta E_{t}\pi_{fh,t+1} && (2.102)\\
\widehat{Y}_{t}-\widehat{Y}^{*}_{t} &= \frac{v}{2}\widehat{d}_{t}-\frac{v}{2}\widehat{d}^{*}_{t}+\widehat{q}_{t}+\frac{2-v}{2}(\widehat{t}_{c,t}-\widehat{t}^{*}_{c,t}) && (2.103)
\end{aligned}$$

Let $\phi_{1,t},\phi_{2,t},\phi_{3,t},\phi_{4,t},\phi_{5,t}$ be lagrangian multipliers on equations (2.99)-(2.103) respectively. First order conditions are:

$$
\begin{aligned}
\widehat{Y}_t - \theta_t &: \widehat{Y}_t - \theta_t + \delta(\phi_{1,t} + \phi_{2,t}) - \phi_{5,t} = 0 && (2.104)\\
\widehat{Y}_t^* - \theta_t^* &: \widehat{Y}_t^* - \theta_t^* + \delta(\phi_{3,t} + \phi_{4,t}) + \phi_{5,t} = 0 && (2.105)\\
\pi_{hh,t} &: \frac{v\lambda}{2\delta}\pi_{hh,t} - \phi_{1,t} + \phi_{1,t-1} = 0 && (2.106)\\
\pi_{hf,t}^* &: \frac{(2-v)\lambda}{2\delta}\pi_{hf,t}^* - \phi_{2,t} + \phi_{2,t-1} = 0 && (2.107)\\
\pi_{ff,t}^* &: \frac{v\lambda}{2\delta}\pi_{ff,t}^* - \phi_{3,t} + \phi_{3,t-1} = 0 && (2.108)\\
\pi_{fh,t} &: \frac{(2-v)\lambda}{2\delta}\pi_{fh,t} - \phi_{4,t} + \phi_{4,t-1} = 0 && (2.109)\\
\widehat{d}_t^* &: \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t}) + \delta(\frac{2-v}{2}\phi_{3,t} - \frac{v}{2}\phi_{4,t} - \frac{v}{2}\phi_{5,t}) = 0 && (2.110)\\
\widehat{d}_t &: \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}_{c,t}^*) + \delta(\frac{2-v}{2}\phi_{1,t} - \frac{v}{2}\phi_{2,t} + \frac{v}{2}\phi_{5,t}) = 0 && (2.111)\\
\widehat{t}_{c,t} &: \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t}) + \delta\frac{2-v}{2}(\phi_{3,t} + \phi_{4,t} + \phi_{5,t}) = 0 && (2.112)\\
\widehat{t}_{c,t}^* &: \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}_{c,t}^*) + \delta\frac{2-v}{2}(\phi_{1,t} + \phi_{2,t} - \phi_{5,t}) = 0 && (2.113)\\
\widehat{s}_{e,t}^* &: \phi_{2,t} = 0 && (2.114)\\
\widehat{s}_{e,t} &: \phi_{4,t} = 0 && (2.115)\\
\widehat{q}_t &: \phi_{5,t} = 0 && (2.116)
\end{aligned}
$$

2.1 Proof of proposition 1

Highlight: Proposition 3 says that tax and subsidy instruments could correct deviation from LOOP and social planner could focus on producer price inflation. global welfare is the efficient one. First step to prove this proposition is show that optimal inflation is zero. With corresponding shadow price $\phi_{i,t} = 0$ and directly we get zero output gap and optimal tax and subsidy rate.

Put (2.114)-(2.115) into (2.107)-(2.109):

$$
\begin{aligned}
\pi_{hf,t}^* &= 0\\
\pi_{fh,t} &= 0
\end{aligned}
$$

From (2.116), we know $\phi_{5,t} = 0$. (2.110)-(2.111) could be simplified as:

$$\begin{aligned} \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{c,t}) + \delta\phi_{3,t} &= 0 & (2.117) \\ \frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^*) + \delta\phi_{1,t} &= 0 & (2.118) \end{aligned}$$

Together with (2.104) and (2.105): we have

$$\begin{aligned} \widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^*) &= 0 & (2.119) \\ \widehat{Y}_t^* - \theta_t^* - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{c,t}) &= 0 & (2.120) \end{aligned}$$

Put these two equations into Phillips curves:

$$\begin{aligned} \pi_{hh,t} &= \delta\left[\widehat{d}_t + \widehat{t}_{c,t}^*\right] + \beta E_t \pi_{hh,t+1} & (2.121) \\ \pi_{ff,t}^* &= \delta\left[\widehat{d}_t^* + \widehat{t}_{c,t}\right] + \beta E_t \pi_{ff,t+1}^* & (2.122) \\ \widehat{t}_{c,t}^* &= \widehat{s}_{e,t} & (2.123) \\ \widehat{t}_{c,t} &= \widehat{s}_{e,t}^* & (2.124) \end{aligned}$$

Let's focus on $\pi_{hh,t}$. Equation (2.117) helps to simplify inflation dynamics as:

$$\pi_{hh,t} = -\frac{2}{v}\delta^2\phi_{1,t} + \beta E_t \pi_{hh,t+1} \tag{2.125}$$

Equation (2.106) is rearranged as:

$$\frac{v\lambda}{2\delta}\pi_{hh,t} = \delta\phi_{1,t} - \delta\phi_{1,t-1} \tag{2.126}$$

The above two equations together solve $\phi_{1,t}$:

$$\begin{aligned} \frac{v\lambda}{2\delta}\pi_{hh,t} &= \delta\phi_{1,t} - \delta\phi_{1,t-1} \\ \frac{2\delta}{v\lambda}\phi_{1,t} - \frac{2\delta}{v\lambda}\phi_{1,t-1} &= -\frac{2}{v}\delta^2\phi_{1,t} + \beta E_t \frac{2\delta}{v\lambda}\phi_{1,t+1} - \beta\frac{2\delta}{v\lambda}\phi_{1,t} \\ \phi_{1,t} - \phi_{1,t-1} &= -\lambda\phi_{1,t} + \beta E_t \phi_{1,t+1} - \beta\phi_{1,t} \end{aligned}$$

Define lag operator, $\phi_{1,t-1} = \phi_{1,t}L$, we have:

$$\phi_{1,t}L^{-1}\left[(1+\lambda\delta+\beta)L - L^2 - \beta\right] = 0$$

Solution of $L^2 - (1+\lambda\delta+\beta)L + \beta$ is $\Delta = \frac{(1+\lambda\delta+\beta)\pm\sqrt{(1+\lambda\delta+\beta)^2-4\beta}}{2}$.

$$\phi_{1,t}L^{-1}(L-\Delta_1)(L-\Delta_2) = 0$$

For these relations to hold at any t, we have $\phi_{1,t} = 0$. Similarly, we have $\phi_{3,t} = 0$.

In sum, the optimal solution is:

$$\widehat{Y}_t - \theta_t = 0,\ \widehat{Y}_t^* - \theta_t^* = 0$$
$$\widehat{d}_t + \widehat{t}^*_{c,t} = 0,\ \widehat{d}_t^* + \widehat{t}_{c,t} = 0$$
$$\widehat{t}^*_{c,t} = \widehat{s}_{e,t},\ \widehat{t}_{c,t} = \widehat{s}^*_{e,t}$$
$$\pi_{hh,t} = 0,\ \pi^*_{ff,t} = 0$$
$$\pi^*_{hf,t} = 0,\ \pi_{fh,t} = 0$$

Optimal solution indicates that central planners should focus on PPI inflation. and use tax policy instruments $\widehat{t}_t^{*c}$, $\widehat{t}_t^c$ to deal with $\widehat{d}_t$ and $\widehat{d}_t^*$. Export subsidy is used to correct the externality of tax. QED.

3 non-cooperation game $\rho = 1$

In non-cooperation game, **Home** loss function:

$$(P2): L_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 - \theta_t \\ +\frac{v(2-v)}{8}(\widehat{d}_t + \widehat{t}^*_{c,t})^2 \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] \\ -(\frac{2-v}{4})\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)^2 + (\frac{2-v}{4})\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)^2 \\ +\frac{(2-v)}{2}\left((\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) - (\widehat{t}^*_{c,t} - \widehat{s}_{e,t})\right) \\ +\frac{2-v}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right)^2 \\ -\frac{2-v}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right)^2 \end{array} \right\} \quad (3.127)$$

Minimize loss function by choosing $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{c,t}, \widehat{s}_{e,t}, \widehat{q}_t\right\}$ given $\left\{\widehat{Y}^*_t - \theta^*_t, \pi^*_{hf,t}, \pi^*_{ff,t}, \widehat{t}^*_{c,t}, \widehat{d}^*_t, \widehat{s}\right.$ subject to

$$\pi_{hh,t} = \delta\left[\widehat{Y}_t - \theta_t + \frac{2-v}{2}\widehat{d}_t + \frac{2-v}{2}\widehat{t}^*_{c,t}\right] + \beta E_t \pi_{hh,t+1} \quad (3.128)$$

$$\pi_{fh,t} = \delta\left[\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}\widehat{d}^*_t + \frac{2-v}{2}\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right] + \beta E_t \pi_{fh,t+1} \quad (3.129)$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \quad (3.130)$$

$\widehat{q}_t$ is term of trade and do not appear in the loss function because $\rho = 1$. $\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}$ are shadow price for constraints (3.128)-(3.130) respectively. First order conditions are:

$$\widehat{Y}_t - \theta_t : (\widehat{Y}_t - \theta_t) - \frac{(2-v)}{2}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\varphi_{1,t} - \varphi_{3,t} = 0 \quad (3.131)$$

$$\pi_{hh,t} : \frac{\lambda v}{2\delta}\pi_{hh,t} + \varphi_{1,t-1} - \varphi_{1,t} = 0 \quad (3.132)$$

$$\pi_{fh,t} : \frac{\lambda(2-v)}{2\delta}\pi_{fh,t} + \varphi_{2,t-1} - \varphi_{2,t} = 0 \quad (3.133)$$

$$\widehat{d}_t : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{1,t} + \frac{v}{2}\varphi_{3,t} = 0 \quad (3.134)$$

$$\widehat{t}_{c,t} : -\frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) - \frac{(2-v)}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) + \frac{(2-v)}{2}(1 + \delta\varphi_{2,t} + \varphi_{3,t}) = 0 \quad (3.135)$$

$$\widehat{s}_{e,t} : -\frac{(2-v)}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) + \frac{(2-v)}{2} = 0 \quad (3.136)$$

$$\widehat{q}_t : \varphi_{3,t} = 0 \quad (3.137)$$

Foreign loss function:

$$
(P2): L_{h,0}^{*} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t^{*} - \theta_t^{*})^2 \\ +\frac{v(2-v)}{8}(\widehat{d}_t^{*} + \widehat{t}_{c,t})^2 \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ +(\frac{2-v}{4})\left(\widehat{t}_{c,t} - \widehat{s}_{e,t}^{*}\right)^2 - (\frac{2-v}{4})\left(\widehat{t}_{c,t}^{*} - \widehat{s}_{e,t}\right)^2 \\ -\frac{(2-v)}{2}\left((\widehat{t}_{c,t} - \widehat{s}_{e,t}^{*}) - (\widehat{t}_t^{*c} - \widehat{s}_{e,t})\right) \\ -\frac{2-v}{4}\left((\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{v}{2}(\widehat{d}_t^{*} + \widehat{t}_{c,t})\right)^2 \\ +\frac{2-v}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^{*})\right)^2 \end{array} \right\} \tag{3.138}
$$

subject to

$$
\pi_{ff,t}^{*} = \delta\left[\widehat{Y}_t^{*} - \theta_t^{*} + \frac{2-v}{2}\widehat{d}_t^{*} + \frac{2-v}{2}\widehat{t}_{c,t}\right] + \beta E_t \pi_{ff,t+1}^{*} \tag{3.139}
$$

$$
\pi_{hf,t}^{*} = \delta\left[\widehat{Y}_t - \theta_t - \frac{v}{2}\widehat{d}_t + \frac{2-v}{2}\widehat{t}_{c,t}^{*} - \widehat{s}_{e,t}\right] + \beta E_t \pi_{hf,t+1}^{*} \tag{3.140}
$$

$$
\widehat{Y}_t - \widehat{Y}_t^{*} = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}_t^{*} + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}_{c,t}^{*}) \tag{3.141}
$$

$\varphi_{1,t}^{*}, \varphi_{2,t}^{*}, \varphi_{3,t}^{*}$ are shadow price for constraints (3.139)-(3.141) respectively. First order conditions are:

$$
\widehat{Y}_t^{*} - \theta_t^{*} : (\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{(2-v)}{2}\left((\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{v}{2}(\widehat{d}_t^{*} + \widehat{t}_{c,t})\right) + \delta\varphi_{1,t}^{*} + \varphi_{3,t}^{*} = 0 \tag{3.142}
$$

$$
\pi_{ff,t}^{*} : \frac{\lambda v}{2\delta}\pi_{ff,t}^{*} + \varphi_{1,t-1}^{*} - \varphi_{1,t}^{*} = 0 \tag{3.143}
$$

$$
\pi_{hf,t}^{*} : \frac{\lambda(2-v)}{2\delta}\pi_{hf,t}^{*} + \varphi_{2,t-1}^{*} - \varphi_{2,t}^{*} = 0 \tag{3.144}
$$

$$
\widehat{d}_t^{*} : \frac{v(2-v)}{4}(\widehat{d}_t^{*} + \widehat{t}_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{v}{2}(\widehat{d}_t^{*} + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{1,t}^{*} - \frac{v}{2}\varphi_{3,t}^{*} = 0 \tag{3.145}
$$

$$
\widehat{t}_{c,t}^{*} : -\frac{(2-v)}{2}\left(\widehat{t}_{c,t}^{*} - \widehat{s}_{e,t}\right) - \frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^{*})\right) + \frac{(2-v)}{2}(1 + \delta\varphi_{2,t}^{*} - \varphi_{3,t}^{*}) = 0 \tag{3.146}
$$

$$
\widehat{s}_{e,t}^{*} : -\frac{(2-v)}{2}\left(\widehat{t}_{c,t} - \widehat{s}_{e,t}^{*}\right) + \frac{(2-v)}{2} = 0 \tag{3.147}
$$

$$
\widehat{q}_t^{*} : \varphi_{3,t}^{*} = 0 \tag{3.148}
$$

3.1 Proof of proposition 2

Highlight: Our first step is to provide the nash equilibrium non-cooperation game. Because the solution returns nonzero inflation rate, non-cooperation game cannot replicate the flexible price allocation. Second part is to find equilibrium with restriction $\widehat{t}^*_{c,t} = \widehat{s}_{e,t}, \widehat{t}_{c,t} = \widehat{s}^*_{e,t}$.

- First, we check whether these choices could be equilibrium solution.

From equation (3.137), we have $\varphi_{3,t} = 0$. (3.135) minus (3.147) :

$$\widehat{t}_{c,t} : -\frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{2,t} + \frac{2-v}{2}\varphi_{3,t} = 0 \tag{3.149}$$

Using the fact that $\varphi_{3,t} = 0$, we simplify Home first order conditions:

$$\begin{aligned}
\widehat{t}_{c,t} : -\frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{2,t} = 0 \\
\Longrightarrow \frac{v}{2}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) = \delta\varphi_{2,t}
\end{aligned} \tag{3.150}$$

$$\begin{aligned}
\widehat{Y}_t - \theta_t : (\widehat{Y}_t - \theta_t) - \frac{(2-v)}{2}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\varphi_{1,t} = 0 \\
\Longrightarrow \frac{v}{2}\left((\widehat{Y}_t - \theta_t) + \frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\varphi_{1,t} = 0
\end{aligned} \tag{3.151}$$

Similarly, we have $\varphi^*_{3,t} = 0$, and

$$\begin{aligned}
\widehat{t}^*_{c,t} : -\frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi^*_{2,t} = 0 \\
\Longrightarrow \frac{v}{2}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) = \delta\varphi^*_{2,t}
\end{aligned} \tag{3.152}$$

$$\begin{aligned}
\widehat{Y}^*_t - \theta^*_t : (\widehat{Y}^*_t - \theta^*_t) - \frac{(2-v)}{2}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\varphi^*_{1,t} = 0 \\
\Longrightarrow \frac{v}{2}\left((\widehat{Y}^*_t - \theta^*_t) + \frac{(2-v)}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\varphi^*_{1,t} = 0
\end{aligned} \tag{3.153}$$

Put these into the NKPCs, we have:

$$\begin{aligned}
\pi_{hh,t} &= -\frac{2\delta^2}{v}\varphi_{1,t} + \beta E_t \pi_{hh,t+1} \\
\pi_{fh,t} &= \frac{2\delta^2}{v}\varphi_{2,t} + \frac{\delta}{2} + \beta E_t \pi_{fh,t+1} \\
\pi^*_{hf,t} &= \frac{2\delta^2}{v}\varphi^*_{2,t} + \frac{\delta}{2} + \beta E_t \pi^*_{hf,t+1} \\
\pi^*_{ff,t} &= -\frac{2\delta^2}{v}\varphi^*_{1,t} + \beta E_t \pi^*_{ff,t+1}
\end{aligned}$$

Similar to preious proof, we can have

$$\begin{aligned}
\varphi_{1,t} &= \varphi^*_{1,t} = 0 \\
\pi_{hh,t} &= \pi^*_{ff,t} = 0 \\
\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) &= 0 \qquad (3.154) \\
(\widehat{Y}^*_t - \theta^*_t) + \frac{(2-v)}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) &= 0 \qquad (3.155)
\end{aligned}$$

With equation (3.144): $\pi^*_{fh,t} = \frac{2\delta}{\lambda(2-v)}[\varphi^*_{2,t} - \varphi^*_{2,t-1}]$ we have

$$\frac{2\delta}{\lambda(2-v)}[\varphi^*_{2,t} - \varphi^*_{2,t-1}] = \frac{2\delta^2}{v}\varphi^*_{2,t} + \frac{\delta}{2} + \beta E_t \frac{2\delta}{\lambda(2-v)}[\varphi^*_{2,t+1} - \varphi^*_{2,t}] \qquad (3.156)$$

Let us assume

$$\varphi^*_{2,t} = A^*_2 \varphi^*_{2,t-1} + B^*_2$$

equation (3.156) could be expressed as

$$\begin{aligned}
\frac{2\delta}{\lambda(2-v)}[(A^*_2 - 1)\varphi^*_{2,t-1} + B^*_2] &= \frac{2\delta^2}{v}A^*_2\varphi^*_{2,t-1} + \frac{2\delta^2}{v}B^*_2 + \frac{\delta}{2} + \beta E_t \frac{2\delta}{\lambda(2-v)}[(A^*_2 - 1)(A^*_2\varphi^*_{2,t-1} + B^*_2) + B^*_2] \qquad (3.157) \\
\frac{2}{\lambda(2-v)}[(A^*_2 - 1)\varphi^*_{2,t-1} + B^*_2] &= \frac{2\delta}{v}A^*_2\varphi^*_{2,t-1} + \frac{2\delta}{v}B^*_2 + \frac{1}{2} + \beta \frac{2}{\lambda(2-v)}[(A^{*2}_2 - A^*_2)\varphi^*_{2,t-1} + A^*_2 B^*_2] \qquad (3.158) \\
\frac{2}{\lambda(2-v)}[(A^*_2 - 1)\varphi^*_{2,t-1} + B^*_2] &= [\beta\frac{2}{\lambda(2-v)}(A^{*2}_2 - A^*_2) + \frac{2\delta}{v}A^*_2]\varphi^*_{2,t-1} + \frac{2\delta}{v}B^*_2 + \frac{1}{2} + \beta\frac{2}{\lambda(2-v)}A^*_2 B^*_2 \qquad (3.159)
\end{aligned}$$

A^*_2 is the solution of following equation:

$$\begin{aligned}
\beta(A^{*2}_2 - A^*_2) + \frac{\lambda(2-v)\delta}{v}A^*_2 - (A^*_2 - 1) &= 0 \\
\beta A^{*2}_2 + [\frac{\lambda(2-v)\delta}{v} - \beta - 1]A^*_2 + 1 &= 0
\end{aligned}$$

$$A_2^* = \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 \pm \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta}}{2\beta}$$

let $\overline{A_2^*}$ be the root whose absolute value is smaller than 1. Further, shadow prices are nonnegative, which means $\overline{A_2^*}$ should be nonnegative as well. This puts restriction on parameters. Suppose there is such parameters. The restrictions are

$$[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta > 0 \tag{3.160}$$

$$0 \leqslant \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 \pm \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta}}{2\beta} < 1 \tag{3.161}$$

If $\frac{\lambda(2-v)\delta}{v} - \beta - 1 > 0$, we cannot find any pamarater set such that $-\frac{\lambda(2-v)\delta}{v} + \beta + 1 \pm \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta} > 0$. Therefore, the only possible situation is that $-\frac{\lambda(2-v)\delta}{v} + \beta + 1 > 0$ and restriction (3.160) can be expressed as:

$$(1 - \sqrt{\beta})^2 > \frac{\lambda(2-v)\delta}{v} \tag{3.162}$$

Two potential solutions are

$$A_{21}^* = \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 + \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta}}{2\beta} \tag{3.163}$$

$$A_{22}^* = \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 - \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta}}{2\beta} \tag{3.164}$$

We can see that

$$A_{21}^* > \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1}{2\beta} > \frac{-(1-\sqrt{\beta})^2 + \beta + 1}{2\beta} > 1$$

Also, we have

$$\begin{aligned} & [\beta + 1 - \frac{\lambda(2-v)\delta}{v}]^2 - 4\beta \\ = \; & [\beta+1]^2 - 4\beta + (\frac{\lambda(2-v)\delta}{v})^2 - 2(\beta+1)\frac{\lambda(2-v)\delta}{v} \\ < \; & [1-\beta]^2 + (\frac{\lambda(2-v)\delta}{v})^2 - 2(1-\beta)\frac{\lambda(2-v)\delta}{v} \\ = \; & [1 - \beta - \frac{\lambda(2-v)\delta}{v}]^2 \end{aligned}$$

Meanwhiel, we have $1-\beta > (1-\sqrt{\beta})^2 > \frac{\lambda(2-v)\delta}{v}$ for any $\beta < 1$.

$$
\begin{aligned}
A_{22}^* &= \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 - \sqrt{[\frac{\lambda(2-v)\delta}{v} - \beta - 1]^2 - 4\beta}}{2\beta} \\
&> \frac{-\frac{\lambda(2-v)\delta}{v} + \beta + 1 - \left(1 - \beta - \frac{\lambda(2-v)\delta}{v}\right)}{2\beta} = 1
\end{aligned}
$$

Neither A_{21}^* nor A_{22}^* satisfy the requirement that $A_2^* < 1$. In other word, we cannot find any nash equilibrium in this game! Next, we consider the case with restriction $\widehat{t}_{c,t} = \widehat{s}_{e,t}^*$ and $\widehat{t}_{c,t}^* = \widehat{s}_{e,t}$.

- we impose the contraint that $\widehat{t}_{c,t} = \widehat{s}^*_{e,t}$ and $\widehat{t}^*_{c,t} = \widehat{s}_{e,t}$.

$$L_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 - \theta_t \\ + \frac{v(2-v)}{8}(\widehat{d}_t + \widehat{t}^*_{c,t})^2 \\ + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] \\ + \frac{2-v}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right)^2 \\ - \frac{2-v}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right)^2 \end{array} \right\} \tag{3.165}$$

$$\pi_{hh,t} = \delta\left[\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right] + \beta E_t \pi_{hh,t+1} \tag{3.166}$$

$$\pi_{fh,t} = \delta\left[\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right] + \beta E_t \pi_{fh,t+1} \tag{3.167}$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \tag{3.168}$$

$\widehat{q}_t$ is term of trade and do not appear in the loss function because $\rho = 1$. First order condition:

$$\widehat{Y}_t - \theta_t : (\widehat{Y}_t - \theta_t) - \frac{(2-v)}{2}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\varphi_{1,t} - \varphi_{3,t} = 0 \tag{3.169}$$

$$\pi_{hh,t} : \frac{\lambda v}{2\delta}\pi_{hh,t} + \varphi_{1,t-1} - \varphi_{1,t} = 0 \tag{3.170}$$

$$\pi_{fh,t} : \frac{\lambda(2-v)}{2\delta}\pi_{fh,t} + \varphi_{2,t-1} - \varphi_{2,t} = 0 \tag{3.171}$$

$$\widehat{d}_t : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{1,t} + \frac{v}{2}\varphi_{3,t} = 0 \tag{3.172}$$

$$\widehat{t}_{c,t} : -\frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{2,t} + \frac{2-v}{2}\varphi_{3,t} = 0 \tag{3.173}$$

$$\widehat{q}_t : \varphi_{3,t} = 0 \tag{3.174}$$

Foreign loss function:

$$L^*_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 - \theta^*_t \\ + \frac{v(2-v)}{8}(\widehat{d}^*_t + \widehat{t}^c_t)^2 \\ + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right] \\ - \frac{2-v}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}^c_t)\right)^2 \\ + \frac{2-v}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^{*c}_t)\right)^2 \end{array} \right\} \tag{3.175}$$

subject to

$$\pi^*_{ff,t} = \delta\left[\widehat{Y}^*_t - \theta^*_t + \frac{2-v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right] + \beta E_t \pi^*_{ff,t+1} \tag{3.176}$$

$$\pi^*_{hf,t} = \delta\left[\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right] + \beta E_t \pi^*_{hf,t+1} \tag{3.177}$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \tag{3.178}$$

First order condition:

$$\widehat{Y}^*_t - \theta^*_t \; : \; (\widehat{Y}^*_t - \theta^*_t) - \frac{(2-v)}{2}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\varphi^*_{1,t} + \varphi^*_{3,t} = 0 \tag{3.179}$$

$$\pi^*_{ff,t} \; : \; \frac{\lambda v}{2\delta}\pi^*_{ff,t} + \varphi^*_{1,t-1} - \varphi^*_{1,t} = 0 \tag{3.180}$$

$$\pi^*_{hf,t} \; : \; \frac{\lambda(2-v)}{2\delta}\pi_{fh,t} + \varphi^*_{2,t-1} - \varphi^*_{2,t} = 0 \tag{3.181}$$

$$\widehat{d}^*_t \; : \; \frac{v(2-v)}{4}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi^*_{1,t} - \frac{v}{2}\varphi^*_{3,t} = 0 \tag{3.182}$$

$$\widehat{t}^*_{c,t} \; : \; -\frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi^*_{2,t} - \frac{2-v}{2}\varphi^*_{3,t} = 0 \tag{3.183}$$

$$\widehat{q}^*_t \; : \; \varphi^*_{3,t} = 0 \tag{3.184}$$

Using the conditions $\varphi_{3,t} = \varphi^*_{3,t} = 0$ and FOCs can be simplified as:

$$\widehat{Y}_t - \theta_t \; : \; (\widehat{Y}_t - \theta_t)\frac{v}{2} + \frac{(2-v)}{2}\frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \delta\varphi_{1,t} = 0 \tag{3.185}$$

$$\widehat{t}_{c,t} \; : \; -\frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{2,t} = 0 \tag{3.186}$$

$$\widehat{Y}^*_t - \theta^*_t \; : \; (\widehat{Y}^*_t - \theta^*_t) - \frac{(2-v)}{2}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\varphi^*_{1,t} = 0 \tag{3.187}$$

$$\widehat{t}^*_{c,t} \; : \; -\frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi^*_{2,t} = 0 \tag{3.188}$$

Use (3.186) to substitute $(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})$ in NKPC $\pi_{hh,t}$:

$$\pi_{hh,t} = -\frac{v}{2}\delta^2\varphi_{1,t} + \beta E_t \pi_{hh,t+1}$$

We replace $\pi_{hh,t}$ using its FOC $\frac{\lambda v}{2\delta}\pi_{hh,t} + \varphi_{1,t-1} - \varphi_{1,t} = 0$. The solution of this difference equation is $\varphi_{1,t} = 0$. Similarly, we have $\varphi_{2,t} = \varphi^*_{1,t} = \varphi^*_{2,t} = 0$

So we have $\pi_{hh,t} = \pi_{fh,t} = \pi^*_{ff,t} = \pi^*_{hf,t} = 0$.

Put the zero shadow prices in FOC, we have $\widehat{Y}_t - \theta_t = 0, \widehat{Y}^*_t - \theta^*_t = 0, \widehat{d}^*_t + \widehat{t}^c_t = 0, \widehat{d}_t + \widehat{t}^{*c}_t = 0$ in the equilibrium. QED.

4 Cooperation game $\rho > 1$

loss function:

$$
\begin{aligned}
L_0 &= L_{h,0} + L^*_{h,0} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c}
\frac{1}{2}(\widehat{Y}_t - \theta_t)^2 + \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 \\
+\frac{v(2-v)}{8}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})^2 \\
+\frac{v(2-v)}{8}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})^2 \\
-\frac{1-\rho}{2}(\widehat{C}^2_t + \widehat{C}^{*2}_t) \\
+\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right]
\end{array} \right\}
\end{aligned}
$$

Market clearing conditions imply:

$$
\begin{aligned}
\widehat{C}_t &= \widehat{Y}_t + \frac{v}{2}\widehat{p}_{hh,t} + \frac{2-v}{2}(\widehat{p}^*_{hf,t} + \widehat{t}^*_{c,t}) + \frac{2-v}{2}\frac{1}{\rho}\widehat{e}_t \\
&= \widehat{Y}_t - \frac{2-v}{2}(\widehat{p}_{fh,t} + \widehat{t}_{c,t}) + \frac{2-v}{2}(\widehat{p}^*_{hf,t} + \widehat{t}^*_{c,t}) + \frac{2-v}{2}\frac{1}{\rho}\widehat{e}_t \\
&= \widehat{Y}_t - \frac{2-v}{2}\widehat{q}_t + \frac{2-v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t}) + \frac{2-v}{2}(\frac{1}{\rho} - 1)\widehat{e}_t \qquad (4.189) \\
\widehat{C}^*_t &= \widehat{Y}^*_t + \frac{v}{2}\widehat{p}^*_{ff,t} + \frac{2-v}{2}(\widehat{p}_{fh,t} + \widehat{t}_{c,t}) - \frac{2-v}{2}\frac{1}{\rho}\widehat{e}_t \\
&= \widehat{Y}^*_t + \frac{2-v}{2}\widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) - \frac{2-v}{2}(\frac{1}{\rho} - 1)\widehat{e}_t \qquad (4.190)
\end{aligned}
$$

We put (4.189) and (4.190) into global loss function:

$$
(P3) : L_0 = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c}
\frac{1}{2}(\widehat{Y}_t - \theta_t)^2 + \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 \\
+\frac{v(2-v)}{8}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t)^2 \\
+\frac{v(2-v)}{8}(\widehat{d}^*_t + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t)^2 \\
-\frac{1-\rho}{2}\left[\widehat{Y}_t - \frac{2-v}{2}\widehat{q}_t + \frac{2-v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t}) + \frac{2-v}{2}(\frac{1}{\rho} - 1)\widehat{e}_t\right]^2 \\
-\frac{1-\rho}{2}\left[\widehat{Y}^*_t + \frac{2-v}{2}\widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) - \frac{2-v}{2}(\frac{1}{\rho} - 1)\widehat{e}_t\right]^2 \\
+\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right]
\end{array} \right\}
$$

The constraints are

$$\pi_{hh,t} = \delta \left[\begin{array}{c} \rho \widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ + \frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array} \right] + \beta E_t \pi_{hh,t+1} \quad (4.191)$$

$$\pi^*_{hf,t} = \delta \left[\begin{array}{c} \rho \widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ - \frac{(2-v)}{2}(\rho - 1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}^*_{c,t} - \widehat{s}_{e,t} \end{array} \right] + \beta E_t \pi^*_{hf,t+1} \quad (4.192)$$

$$\pi^*_{ff,t} = \delta \left[\begin{array}{c} \rho \widehat{Y}^*_t - \theta^*_t + \frac{2-v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ + \frac{(2-v)(1-\rho)}{2}(\widehat{t}^*_{c,t} - \widehat{t}^c_t) \end{array} \right] + \beta E_t \pi^*_{ff,t+1} \quad (4.193)$$

$$\pi_{fh,t} = \delta \left[\begin{array}{c} \rho \widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ + \frac{2-v}{2}(\rho - 1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t} - \widehat{s}^*_{e,t} \end{array} \right] + \beta E_t \pi_{fh,t+1} \quad (4.194)$$

$$\begin{aligned} \widehat{Y}_t - \widehat{Y}^*_t &= -\frac{v}{2}\widehat{p}_{hh,t} - \frac{2-v}{2}(\widehat{p}^*_{hf,t}) + \frac{v-1}{\rho}\widehat{e}_t + \frac{v}{2}\widehat{p}^*_{ff,t} + \frac{2-v}{2}\widehat{p}_{fh,t} + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \\ &= \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \end{aligned} \quad (4.195)$$

$$\widehat{e}_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t - (1-v)\widehat{q}_t - \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \quad (4.196)$$

Let $\xi_{1,t}, \xi_{2,t}, \xi_{3,t}, \xi_{4,t}, \xi_{5,t}, \xi_{6,t}$ be lagrangian multipliers on equation (4.191)-(4.196) respectively. Optimal choices are $\left\{\widehat{Y}_t - \theta_t, \widehat{Y}^*_t - \theta^*_t, \pi_{hh,t}, \pi^*_{hf,t}, \pi^*_{ff,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{d}^*_t, \widehat{t}_{c,t}, \widehat{t}^*_{c,t}, \widehat{s}^*_{e,t}, \widehat{s}_{e,t}, , \widehat{q}_t, \widehat{e}_t\right\}$

First order conditions are:

$$\widehat{Y}_t - \theta_t : \widehat{Y}_t - \theta_t - (1-\rho)\widehat{C}_t + \delta\rho(\xi_{1,t} + \xi_{2,t}) - \xi_{5,t} = 0 \tag{4.197}$$

$$\widehat{Y}_t^* - \theta_t^* : \widehat{Y}_t^* - \theta_t^* - (1-\rho)\widehat{C}_t^* + \delta\rho(\xi_{3,t} + \xi_{4,t}) + \xi_{5,t} = 0 \tag{4.198}$$

$$\pi_{hh,t} : \frac{v\lambda}{2\delta}\pi_{hh,t} - \xi_{1,t} + \xi_{1,t-1} = 0 \tag{4.199}$$

$$\pi_{hf,t}^* : \frac{(2-v)\lambda}{2\delta}\pi_{hf,t}^* - \xi_{2,t} + \xi_{2,t-1} = 0 \tag{4.200}$$

$$\pi_{ff,t}^* : \frac{v\lambda}{2\delta}\pi_{ff,t}^* - \xi_{3,t} + \xi_{3,t-1} = 0 \tag{4.201}$$

$$\pi_{fh,t} : \frac{(2-v)\lambda}{2\delta}\pi_{fh,t} - \xi_{4,t} + \xi_{4,t-1} = 0 \tag{4.202}$$

$$\widehat{d}_t : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}_{c,t}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) + \delta\frac{(2-v)}{2}\xi_{1,t} - \delta\frac{v}{2}\xi_{2,t} + \frac{v}{2}\xi_{5,t} + \frac{v}{2}\xi_{6,t} = 0 \tag{4.203}$$

$$\widehat{d}_t^* : \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t) + \delta\frac{(2-v)}{2}\xi_{3,t} - \delta\frac{v}{2}\xi_{4,t} - \frac{v}{2}\xi_{5,t} - \frac{v}{2}\xi_{6,t} = 0 \tag{4.204}$$

$$\begin{aligned}\widehat{t}_{c,t} : \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t) + \frac{2-v}{2}(1-\rho)(\widehat{C}_t - \widehat{C}_t^*) \\ +\delta\frac{(2-v)(1-\rho)}{2}(\xi_{1,t} + \xi_{2,t}) + \delta\frac{(2-v)}{2}\xi_{3,t} - \delta\frac{(2-v)(1-\rho)}{2}(\xi_{3,t} + \xi_{4,t}) \\ +\delta\frac{2-v}{2}\xi_{4,t} + \frac{2-v}{2}\xi_{5,t} - \frac{2-v}{2}\xi_{6,t} = 0\end{aligned} \tag{4.205}$$

$$\begin{aligned}\widehat{t}_{c,t}^* : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}_{c,t}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) - \frac{2-v}{2}(1-\rho)(\widehat{C}_t - \widehat{C}_t^*) \\ +\delta\frac{(2-v)(1-\rho)}{2}(\xi_{3,t} + \xi_{4,t}) + \delta\frac{(2-v)}{2}\xi_{1,t} - \delta\frac{(2-v)(1-\rho)}{2}(\xi_{1,t} + \xi_{2,t}) \\ +\delta\frac{2-v}{2}\xi_{2,t} - \frac{2-v}{2}\xi_{5,t} + \frac{2-v}{2}\xi_{6,t} = 0\end{aligned} \tag{4.206}$$

$$\widehat{s}_{e,t}^* : \xi_{2,t} = 0 \tag{4.207}$$

$$\widehat{s}_{e,t} : \xi_{4,t} = 0 \tag{4.208}$$

$$\widehat{q}_t : \frac{2-v}{2}(1-\rho)(\widehat{C}_t - \widehat{C}_t^*) + \delta\frac{2-v}{2}(1-\rho)(\xi_{1,t} + \xi_{2,t}) - \delta\frac{2-v}{2}(1-\rho)(\xi_{3,t} + \xi_{4,t}) + \xi_{5,t} - (1-v)\xi_{6,t} = 0 \tag{4.209}$$

$$\begin{aligned}\widehat{e}_t : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}_{c,t}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t)(\frac{1}{\rho} - 1) - \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t)(\frac{1}{\rho} - 1) \\ -(1-\rho)\left[\widehat{C}_t - \widehat{C}_t^*\right]\frac{2-v}{2}(\frac{1}{\rho} - 1) \\ +\delta\frac{2-v}{2}(1-\rho)(\xi_{1,t} + \xi_{2,t}) - \delta\frac{2-v}{2}(1-\rho)(\xi_{3,t} + \xi_{4,t}) \\ +(v-1)(\frac{1}{\rho} - 1)\xi_{5,t} - \xi_{6,t} = 0\end{aligned} \tag{4.210}$$

4.1 Game solution with $\rho > 1$

Highlight: Method to prove optimal choince under case with $\rho > 1$ is similar with previous case $\rho = 1$ except that equations here could be more tedious.

Optimal choices of subsidy rate (4.207) and (4.208) imply $\pi^*_{hf,t} = 0$ and $\pi_{fh,t} = 0$. Replacing the terms in (4.210) by using From (4.203)-(4.203) and (4.209), we have:

$$\begin{aligned}
\widehat{e}_t : -\delta\frac{(2-v)}{2}(\frac{1}{\rho}-1)\xi_{1,t} - (\frac{1}{\rho}-1)\frac{v}{2}\xi_{5,t} - (\frac{1}{\rho}-1)\frac{v}{2}\xi_{6,t} \qquad (4.211)\\
+\delta\frac{(2-v)}{2}(\frac{1}{\rho}-1)\xi_{3,t} - (\frac{1}{\rho}-1)\frac{v}{2}\xi_{5,t} - (\frac{1}{\rho}-1)\frac{v}{2}\xi_{6,t}\\
\delta\frac{2-v}{2}(1-\rho)(\frac{1}{\rho}-1)\xi_{1,t} - \delta\frac{2-v}{2}(1-\rho)(\frac{1}{\rho}-1)\xi_{3,t} + (\frac{1}{\rho}-1)\xi_{5,t} - (1-v)(\frac{1}{\rho}-1)\xi_{6,t}\\
+\delta\frac{2-v}{2}(1-\rho)(\xi_{1,t}) - \delta\frac{2-v}{2}(1-\rho)(\xi_{3,t})\\
+(v-1)(\frac{1}{\rho}-1)\xi_{5,t} - \xi_{6,t} = 0
\end{aligned}$$

Further simplify the above equation, we get:

$$\xi_{6,t} = 0 \qquad (4.212)$$

From (4.197) and (4.189), we have

$$\rho\widehat{Y}_t - \theta_t + (1-\rho)\frac{2-v}{2}\left[(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{q}_t + \widehat{e}_t\right] = (\frac{1}{\rho}-1)\frac{2-v}{2}\widehat{e}_t - \delta\rho\xi_{1,t} + \xi_{5,t}$$

We define

$$\begin{aligned}
M_t &= \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) + \frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t})\\
&= (\frac{1}{\rho}-1)\frac{2-v}{2}\widehat{e}_t - \delta\rho\xi_{1,t} + \xi_{5,t} + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) \qquad (4.213)
\end{aligned}$$

so

$$\pi_{hh,t} = \delta\left[M_t\right] + \beta E_t\pi_{hh,t+1}$$

Equation (4.203) could be rearranged as:

$$\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) = -\frac{(2-v)}{2}(\frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) - \delta\frac{(2-v)}{v}\xi_{1,t} - \xi_{5,t} - \xi_{6,t}$$

Together with equation (4.213), we have

$$\begin{aligned} M_t &= (\frac{1}{\rho}-1)\frac{2-v}{2}\widehat{e}_t - \delta\rho\xi_{1,t} + \xi_{5,t} - \frac{(2-v)}{2}(\frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) - \delta\frac{(2-v)}{v}\xi_{1,t} - \xi_{5,t} \\ &= -\delta\left[\rho + \frac{(2-v)}{v}\right]\xi_{1,t} \end{aligned}$$

Substitute M_t in Phillips curve and replace $\pi_{hh,t}$ using equation (4.199) :

$$\begin{aligned} \frac{2\delta}{v\lambda}\xi_{1,t} - \frac{2\delta}{v\lambda}\xi_{1,t-1} &= -\delta^2\left[\rho + \frac{(2-v)}{v}\right]\xi_{1,t} + \beta E_t\frac{2\delta}{v\lambda}\xi_{1,t+1} - \beta\frac{2\delta}{v\lambda}\xi_{1,t} \\ \xi_{1,t} - \xi_{1,t-1} &= -\delta\frac{v\lambda}{2}\left[\rho + \frac{(2-v)}{v}\right]\xi_{1,t} + \beta E_t\xi_{1,t+1} - \beta\xi_{1,t} \end{aligned}$$

Use lag operator, $\xi_{1,t-1} = \xi_{1,t}L$, we have:

$$\xi_{1,t}L^{-1}\left[\left(1 + \delta\frac{v\lambda}{2}(\rho + \frac{(2-v)}{v}) + \beta\right)L - L^2 - \beta\right] = 0$$

Solutions of $L^2 - \left(1 + \delta\frac{v\lambda}{2}(\rho + \frac{(2-v)}{v}) + \beta\right)L + \beta$ are $\Delta = \frac{(1+\delta\frac{v\lambda}{2}(\rho+\frac{(2-v)}{v})+\beta) \pm \sqrt{(1+\delta\frac{v\lambda}{2}(\rho+\frac{(2-v)}{v})+\beta)^2 - 4\beta}}{2}$

$$\xi_{1,t}L^{-1}(L-\Delta_1)(L-\Delta_2) = 0$$

For these relations to hold at any t, we have $\xi_{1,t} = 0$.Similarly, we have $\xi_{3,t} = 0$.
Optimal choice of $\widehat{d}_t$ (4.203) minus Optimal choice of $\widehat{t}^*_{c,t}$ (4.206):

$$\xi_{5,t} + \frac{2-v}{2}(1-\rho)(\widehat{C}_t - \widehat{C}^*_t) = 0$$

Together with $\widehat{C}_t - \widehat{C}^*_t = \frac{1}{\rho}\widehat{e}_t$ we get:

$$\xi_{5,t} = \frac{2-v}{2}(1-\frac{1}{\rho})\widehat{e}_t$$

Together with (4.203):

$$\widehat{d}_t + \widehat{t}^*_{c,t} = 0$$

Similarly, we have $\widehat{d}^*_t + \widehat{t}_{c,t} = 0$.
NKPCs implies:

$$\widehat{t}^*_{c,t} = \widehat{s}_{e,t}, \widehat{t}_{c,t} = \widehat{s}^*_{e,t}$$

(4.195) could be simplified:

$$\widehat{Y}_t - \widehat{Y}_t^* = \widehat{q}_t + \widehat{t}_{c,t} - \widehat{t}_{c,t}^* + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \tag{4.214}$$

(4.189) minus (4.190):

$$\begin{aligned} \frac{1}{\rho}\widehat{e}_t &= \widehat{Y}_t - \widehat{Y}_t^* - (2-v)(\widehat{q}_t + \widehat{t}_{c,t} - \widehat{t}_{c,t}^*) + (2-v)(\frac{1}{\rho} - 1)\widehat{e}_t \\ &= \widehat{Y}_t - \widehat{Y}_t^* - (2-v)(\widehat{q}_t + \widehat{t}_{c,t} - \widehat{t}_{c,t}^* + \frac{v-1}{\rho}\widehat{e}_t) + (2-v)(\frac{1}{\rho} - 1 + \frac{v-1}{\rho})\widehat{e}_t \end{aligned} \tag{4.215}$$

Substitute (4.214) into (4.215):

$$(\widehat{Y}_t - \widehat{Y}_t^*)(v-1) = \left[\frac{1}{\rho} - (2-v)v(\frac{1}{\rho} - 1)\right]\widehat{e}_t \tag{4.216}$$

$\pi_{hh,t} = \pi_{ff,t}^* = 0$ imply:

$$\rho(\widehat{Y}_t - \widehat{Y}_t^*) - \theta_t + \theta_t^* + (1-\rho)(2-v)\left[\widehat{Y}_t - \widehat{Y}_t^* - (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t + \widehat{e}_t\right] = 0 \tag{4.217}$$

The above two equation help to get expression

$$\frac{1}{(v-1)}\left[\rho + (1-\rho)(2-v)\right]\left[\frac{1}{\rho} - (2-v)v(\frac{1}{\rho} - 1)\widehat{e}_t\right]\widehat{e}_t - (1-\rho)(2-v)\left[v + \frac{1}{\rho} - \frac{v}{\rho}\right]\widehat{e}_t = \theta_t - \theta_t^* \tag{4.218}$$

Rearranging the terms, we get:

$$\widehat{e}_t = (v-1)(\theta_t - \theta_t^*)$$

Replacing expression of $\widehat{e}_t$ in FOC (4.197) and we get the optimal output gap:

$$\rho\widehat{Y}_t - \theta_t + (1-\rho)\frac{(2-v)v}{2}(\theta_t - \theta_t^*) = 0$$

Similarly, we have $\rho\widehat{Y}_t^* - \theta_t^* - (1-\rho)\frac{(2-v)v}{2}(\theta_t - \theta_t^*) = 0$

The solution is different from case when $\rho = 1$. In sum, the optimal solution is:

$$\rho\widehat{Y}_t - \theta_t + (1-\rho)\frac{(2-v)v}{2}(\theta_t - \theta_t^*) = 0$$
$$\rho\widehat{Y}_t^* - \theta_t^* - (1-\rho)\frac{(2-v)v}{2}(\theta_t - \theta_t^*) = 0$$
$$\widehat{d}_t + \widehat{t}_{c,t}^* = 0,\ \widehat{d}_t^* + \widehat{t}_{c,t} = 0$$
$$\widehat{t}_{c,t}^* = \widehat{s}_{e,t},\ \widehat{t}_{c,t} = \widehat{s}_{e,t}^*$$
$$\pi_{hh,t} = 0,\ \pi_{ff,t}^* = 0$$
$$\pi_{hf,t}^* = 0,\ \pi_{fh,t} = 0$$

QED.

4.2 Welfare loss without home bias $v = 1$

we have**:**

$$\widehat{Y}_t - \theta_t = \frac{(1-\rho)}{2\rho}(\theta_t + \theta_t^*)$$
$$\widehat{Y}_t^* - \theta_t^* = \frac{(1-\rho)}{2\rho}(\theta_t + \theta_t^*)$$
$$\widehat{e}_t = 0,$$
$$\widehat{C}_t = \widehat{C}_t^* = \frac{1}{2\rho}(\theta_t + \theta_t^*)$$
$$\widehat{d}_t + \widehat{t}_{c,t}^* = 0,\ \widehat{d}_t^* + \widehat{t}_{c,t} = 0$$
$$\widehat{t}_{c,t}^* = \widehat{s}_{e,t},\ \widehat{t}_{c,t} = \widehat{s}_{e,t}^*$$
$$\pi_{hh,t} = 0,\ \pi_{ff,t}^* = 0$$
$$\pi_{hf,t}^* = 0,\ \pi_{fh,t} = 0$$

Put the expression of policy and consumption into loss function, we have

$$
\begin{aligned}
L &= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 + \frac{1}{2}(\widehat{Y}_t^* - \theta_t^*)^2 \\ + \frac{v(2-v)}{8}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)^2 \\ + \frac{v(2-v)}{8}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})^2 \\ - \frac{1-\rho}{2}(\widehat{C}_t^2 + \widehat{C}_t^{*2}) \\ + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{hh,t}^2 + \frac{(2-v)}{2}\pi_{fh,t}^2\right] + \frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \end{array} \right\} \\
&= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \frac{(1-\rho)^2}{4\rho^2}(\theta_t + \theta_t^*)^2 - \frac{1-\rho}{4\rho^2}(\theta_t + \theta_t^*))^2 \right\}
\end{aligned}
$$

5 Non-cooperation game, $\rho > 1$ with restriction $\widehat{t}^*_{c,t} = \widehat{s}_{e,t}, \widehat{t}_{c,t} = \widehat{s}^*_{e,t}$

Home loss function:

$$(P4) : L_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\ +\frac{v(2-v)}{8}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})^2 \\ +\frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] \\ +\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right] \\ -\frac{1-\rho}{2}\widehat{C}^2_t + \frac{(2-v)}{4\rho}(1-\rho)^2\left((\widehat{C}^*_t)^2 - \widehat{C}^2_t\right) \\ +\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ -\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \end{array} \right\} \tag{5.219}$$

The authority minimizes loss function by choosing $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{c,t}, \widehat{q}_t, \widehat{e}_t\right\}$ given $\{\widehat{Y}^*_t - \theta^*_t, \pi^*_{hf,t}, \pi^*_{ff,t}, \widehat{t}^*_{c,t}, \widehat{d}^*_t\}$, subject to following constraint, shadow prices are $\{\varpi_{1,t}, \varpi_{2,t}, \varpi_{3,t}, , \varpi_{4,t}\}$:

$$\pi_{hh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array} \right] + \beta E_t \pi_{hh,t+1} \tag{5.220}$$

$$\pi_{fh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ +\frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array} \right] + \beta E_t \pi_{fh,t+1} \tag{5.221}$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \tag{5.222}$$

$$\widehat{e}_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t - (1-v)\widehat{q}_t - \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \tag{5.223}$$

Foreign loss function:

$$(P4): L^*_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t)^2 \\ +\frac{v(2-v)}{8}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})^2 \\ +\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] + \frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right] \\ -\frac{1-\rho}{2}\widehat{C}^{*2}_t + \frac{(2-v)}{4\rho}(1-\rho)^2\left(\widehat{C}^2_t - (\widehat{C}^*_t)^2\right) \\ -\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \end{array} \right\} \quad (5.224)$$

Minimize loss function by choosing $\left\{\widehat{Y}^*_t - \theta^*_t, \pi^*_{hf,t}, \pi^*_{ff,t}, \widehat{t}^*_{c,t}, \widehat{d}^*_t, \widehat{q}^*_t, \widehat{e}^*_t\right\}$ given $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{c,t}\right\}$, subject to following constraint, shadow prices are $\{\varpi^*_{1,t}, \varpi^*_{2,t}, \varpi^*_{3,t}, \varpi^*_{4,t}\}$:

$$\pi^*_{hf,t} = \delta\left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ -\frac{(2-v)}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array}\right] + \beta E_t \pi^*_{hf,t+1} \quad (5.225)$$

$$\pi^*_{ff,t} = \delta\left[\begin{array}{c} \rho\widehat{Y}^*_t - \theta^*_t + \frac{2-v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}^*_{c,t} - \widehat{t}^c_t) \end{array}\right] + \beta E_t \pi^*_{ff,t+1} \quad (5.226)$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \quad (5.227)$$

$$\widehat{e}_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t - (1-v)\widehat{q}_t - \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \quad (5.228)$$

We define

$$
\begin{aligned}
M_1 &= (\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t}) \\
M_2 &= (\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t}) \\
M_3 &= (\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t}) \\
M_4 &= (\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})
\end{aligned}
$$

The first order conditions of $\widehat{Y}_t - \theta_t$:

$$
\begin{gathered}
(\widehat{Y}_t - \theta_t) - (1-\rho)\widehat{C}_t + \frac{v(2-v)}{8}\frac{1-\rho}{\rho}2M_1 \\
-\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}2M_4 + \delta\rho\varpi_{1,t} - \varpi_{3,t} - \frac{(2-v)}{2\rho}(1-\rho)^2\widehat{C}_t = 0
\end{gathered} \tag{5.229}
$$

The first order conditions of $\pi_{hh,t}$ and $\pi_{fh,t}$:

$$
\pi_{hh,t} : \frac{\lambda(1+\frac{\alpha}{\rho})v}{4\delta}\pi_{hh,t} + \varpi_{1,t-1} - \varpi_{1,t} = 0 \tag{5.230}
$$

$$
\pi_{fh,t} : \frac{\lambda(1+\frac{\alpha}{\rho})(2-v)}{4\delta}\pi_{fh,t} + \varpi_{2,t-1} - \varpi_{2,t} = 0 \tag{5.231}
$$

The first order conditions of $\widehat{d}_t$:

$$
\begin{gathered}
\frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) + \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_1 \\
+\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}vM_4 + \delta\frac{2-v}{2}\varpi_{1,t} + \frac{v}{2}\varpi_{3,t} + \frac{v}{2}\varpi_{4,t} = 0
\end{gathered} \tag{5.232}
$$

The first order conditions of $\widehat{t}_{c,t}$:

$$
\begin{gathered}
((1-\rho))\,\widehat{C}_t\frac{(2-v)}{2} + \frac{(2-v)}{4\rho}(1-\rho)^2(2-v)\left(\widehat{C}^*_t + \widehat{C}_t\right) \\
-\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}M_2 v - \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_3 \\
+\delta\frac{(2-v)(1-\rho)}{2}\varpi_{1,t} + \delta\frac{(2-v)(\rho-1)}{2}\varpi_{2,t} - \delta\frac{v}{2}\varpi_{2,t} + \frac{2-v}{2}\varpi_{3,t} - \frac{2-v}{2}\varpi_{4,t} = 0
\end{gathered} \tag{5.233}
$$

The first order conditions of $\widehat{e}_t$:

$$
\begin{aligned}
0 = {} & \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t)\frac{1-\rho}{\rho} \\
& -(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}\widehat{C}_t \\
& -\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\frac{1-\rho}{\rho}\left(-\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}_t\right) \\
& +\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_1 + \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_2 \\
& +\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_3 + \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_4 \\
& +\frac{(2-v)(1-\rho)}{2}\delta\varpi_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi_{2,t} + \varpi_{3,t}(\frac{1}{\rho}-1)(v-1) - \varpi_{4,t}
\end{aligned}
\tag{5.234}
$$

First order condition of $\widehat{q}_t$:

$$
\begin{aligned}
0 \quad = \quad & (1-\rho)\frac{(2-v)}{2}\widehat{C}_t \\
& +\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\left(-\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}_t\right) \\
& +\frac{(2-v)(1-\rho)}{2}\delta\varpi_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi_{2,t} \\
& +\varpi_{3,t} - (1-v)\varpi_{4,t}
\end{aligned}
\tag{5.235}
$$

FOC of foreign:

The first order condition of $\widehat{Y}_t^* - \theta_t^*$:

$$\widehat{Y}_t^* - \theta_t^* : (\widehat{Y}_t^* - \theta_t^*) - (1-\rho)\widehat{C}_t^* + \frac{v(2-v)}{8}\frac{1-\rho}{\rho}2M_3 \tag{5.236}$$
$$-\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}2M_2 + \delta\rho\varpi_{2,t}^* + \varpi_{3,t}^* - \frac{(2-v)}{2\rho}(1-\rho)^2\widehat{C}_t^* = 0$$

The first order condition of $\pi_{hf,t}^*$ and $\pi_{ff,t}^*$:

$$\pi_{hf,t}^* : \frac{\lambda(1+\frac{\alpha}{\rho})(2-v)}{4\delta}\pi_{hf,t}^* + \varpi_{1,t-1}^* - \varpi_{1,t}^* = 0 \tag{5.237}$$
$$\pi_{ff,t}^* : \frac{\lambda(1+\frac{\alpha}{\rho})v}{4\delta}\pi_{ff,t}^* + \varpi_{2,t-1}^* - \varpi_{2,t}^* = 0 \tag{5.238}$$

The first order condition of $\widehat{d}_t^*$:

$$\widehat{d}_t^* : \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t) + \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_3 \tag{5.239}$$
$$+\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}vM_2 + \delta\frac{2-v}{2}\varpi_{2,t}^* - \frac{v}{2}\varpi_{3,t}^* - \frac{v}{2}\varpi_{4,t}^* = 0$$

The first order condition of $\widehat{t}_{c,t}^*$:

$$\widehat{t}_{c,t}^* : (1-\rho)\widehat{C}_t^*\frac{(2-v)}{2} + \frac{(2-v)}{4\rho}(1-\rho)^2(2-v)\left(\widehat{C}_t^* + \widehat{C}_t\right) \tag{5.240}$$
$$-\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}vM_4 - \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_1$$
$$+\delta\frac{(2-v)(1-\rho)}{2}\varpi_{2,t}^* - \delta\frac{(2-v)(1-\rho)}{2}\varpi_{1,t}^* - \delta\frac{v}{2}\varpi_{1,t}^* - \frac{2-v}{2}\varpi_{3,t}^* + \frac{2-v}{2}\varpi_{4,t}^* = 0$$

The first order conditions of $\widehat{e}_t$:

$$0 = -\frac{v(2-v)}{4}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\frac{1-\rho}{\rho} \tag{5.241}$$
$$+(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}\widehat{C}^*_t$$
$$+\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\frac{1-\rho}{\rho}\left(\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}^*_t\right)$$
$$-\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_1 - \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_2$$
$$-\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_3 - \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_4$$
$$+\frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{2,t} + \varpi^*_{3,t}(\frac{1}{\rho} - 1)(v-1) - \varpi^*_{4,t}$$

First order condition of $\widehat{q}^*_t$:

$$\begin{aligned} 0 \quad = \quad & (1-\rho)\frac{(2-v)}{2}\widehat{C}^*_t \\ & +\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\left(\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}^*_t\right) \\ & -\frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{1,t} + \frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{2,t} \\ & -\varpi^*_{3,t} + (1-v)\varpi^*_{4,t} \end{aligned} \tag{5.242}$$

5.1 Game solution

- **Highlight:** Method to prove is similar as in previous case except that notations could be tedious here.

From equation (5.232), we have:

$$\varpi_{3,t} = -\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) - \frac{(2-v)^2}{4}\frac{1-\rho}{\rho}M_1$$
$$-\frac{v(2-v)}{4}\frac{\rho - 1 + \frac{2}{v}}{\rho}M_4 - \delta\frac{2-v}{v}\varpi_{1,t} - \varpi_{4,t} \tag{5.243}$$

Replace $\varpi_{3,t}$ in equation (5.233) with the above equation (5.243):

$$((1-\rho))\,\widehat{C}_t\frac{(2-v)}{2} + \frac{(2-v)}{4\rho}(1-\rho)^2(2-v)\left(\widehat{C}^*_t + \widehat{C}_t\right) \tag{5.244}$$
$$-\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}M_2 v - \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_3$$
$$+\delta\frac{(2-v)(1-\rho)}{2}\varpi_{1,t} + \delta\frac{(2-v)(\rho-1)}{2}\varpi_{2,t} - \delta\frac{v}{2}\varpi_{2,t} - \frac{2-v}{2}\varpi_{4,t}$$
$$-\frac{2-v}{2}\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t)$$
$$-\frac{2-v}{2}\frac{(2-v)^2}{4}\frac{1-\rho}{\rho}M_1$$
$$-\frac{2-v}{2}\frac{v(2-v)}{4}\frac{\rho - 1 + \frac{2}{v}}{\rho}M_4 - \frac{2-v}{2}\frac{2-v}{v}\delta\varpi_{1,t} - \frac{2-v}{2}\varpi_{4,t} = 0 \tag{5.245}$$

Replace $\varpi_{3,t}$ in equation (5.229) with the above equation (5.243):

$$\begin{gathered}(\widehat{Y}_t-\theta_t)\\ -(1-\rho)(\widehat{Y}_t-\frac{2-v}{2}\widehat{q}_t+\frac{2-v}{2}(\widehat{t}^*_{c,t}-\widehat{t}_{c,t})+\frac{2-v}{2}(\frac{1}{\rho}-1)\widehat{e}_t)\\ +\frac{v(2-v)}{4}\frac{1-\rho}{\rho}((\widehat{Y}_t-\theta_t)+\frac{2-v}{2}(\widehat{d}_t+\widehat{t}^*_{c,t}+\frac{1}{\rho}\widehat{e}_t-\widehat{e}_t))\\ +\delta\rho\varpi_{1,t}-\frac{(2-v)}{2\rho}(1-\rho)^2(\widehat{Y}_t-\frac{2-v}{2}\widehat{q}_t+\frac{2-v}{2}(\widehat{t}^*_{c,t}-\widehat{t}_{c,t})+\frac{2-v}{2}(\frac{1}{\rho}-1)\widehat{e}_t)\\ +\frac{(2-v)}{2}(\widehat{d}_t+\widehat{t}^*_{c,t}+\frac{1}{\rho}\widehat{e}_t-\widehat{e}_t)\\ +\frac{(2-v)^2}{4}\frac{1-\rho}{\rho}((\widehat{Y}_t-\theta_t)+\frac{2-v}{2}(\widehat{d}_t+\widehat{t}^*_{c,t}+\frac{1}{\rho}\widehat{e}_t-\widehat{e}_t))\\ +\delta\frac{2-v}{v}\varpi_{1,t}+\varpi_{4,t}=0\end{gathered}$$

Rearranging terms, we have expression for $\varpi_{1,t}$:

$$\begin{gathered}\frac{1}{2}(\widehat{Y}_t-\theta_t)\\ +\frac{1}{v}\delta\varpi_{1,t}+\frac{1}{2-v+v\rho}\varpi_{4,t}\\ +\frac{1}{2\rho}\left(\rho-1\right)(\theta_t-\frac{2-v}{2}\widehat{q}_t+\frac{2-v}{2}(\widehat{t}^*_{c,t}-\widehat{t}_{c,t}))\\ -\frac{1}{4\rho}\left(v-2\right)(\widehat{d}_t+\widehat{t}^*_{c,t})\\ +\frac{1}{4\rho}\left(\rho-1\right)\left(v-2\right)\widehat{e}_t=0\end{gathered}\tag{5.246}$$

$$\begin{gathered}\rho(\widehat{Y}_t-\theta_t)\\ +\frac{2\rho}{v}\delta\varpi_{1,t}+\frac{2\rho}{2-v+v\rho}\varpi_{4,t}\\ +\left(\rho-1\right)(\theta_t-\frac{2-v}{2}\widehat{q}_t+\frac{2-v}{2}(\widehat{t}^*_{c,t}-\widehat{t}_{c,t}))\\ -\frac{1}{2}\left(v-2\right)(\widehat{d}_t+\widehat{t}^*_{c,t})\\ +\frac{1}{2}\left(\rho-1\right)\left(v-2\right)\widehat{e}_t=0\end{gathered}\tag{5.247}$$

We can easily find that:

$$\rho\widehat{Y}_t-\theta_t+\frac{2-v}{2}(\widehat{d}_t+\widehat{t}^*_{c,t})+\frac{2-v}{2}(1-\rho)(\widehat{q}_t+\widehat{e}_t)+\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t}-\widehat{t}^*_{c,t})$$
$$=\quad -\frac{2\rho}{v}\delta\varpi_{1,t}-\frac{2\rho}{2-v+v\rho}\varpi_{4,t} \tag{5.248}$$

This is the first key equation for our solution. Therefore:

$$\pi_{hh,t}=\delta\left[-\frac{2\rho}{v}\delta\varpi_{1,t}-\frac{2\rho}{2-v+v\rho}\varpi_{4,t}\right]+\beta E_t\pi_{hh,t+1} \tag{5.249}$$

- By combining the FOC of $\widehat{t}_t$ and $\widehat{e}_t$, and employing equation (5.243), we have:

$$\delta\left(\rho-1\right)\varpi_{2,t}=\varpi_{4,t} \tag{5.250}$$

- Replace $\varpi_{2,t}$ in the the first order conditions of $\widehat{q}_t$, equation (5.235):

$$\begin{aligned}0 \quad = \quad & (1-\rho)\frac{(2-v)}{2}\widehat{C}_t \\ & +\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\left(-\frac{1}{\rho}\widehat{e}_t+2\widehat{C}_t\right) \\ & +\frac{(2-v)(1-\rho)}{2}\delta\varpi_{1,t}+\frac{v}{2}\varpi_{4,t}+\varpi_{3,t}\end{aligned} \tag{5.251}$$

By replacing $\varpi_{3,t}$ in the above equation using equation (5.243), we have:

$$\begin{aligned}0 \quad = \quad & (1-\rho)\frac{(2-v)}{2}\widehat{C}_t \\ & +\frac{(2-v)}{2\rho}(1-\rho)^2\frac{(2-v)}{2}\left(-\frac{1}{\rho}\widehat{e}_t+2\widehat{C}_t\right) \\ & +\frac{(2-v)(1-\rho)}{2}\delta\varpi_{1,t}+\frac{v}{2}\varpi_{4,t} \\ & -\frac{(2-v)}{2}(\widehat{d}_t+\widehat{t}^*_{c,t}+\frac{1}{\rho}\widehat{e}_t-\widehat{e}_t) \\ & -\frac{(2-v)^2}{4}\frac{1-\rho}{\rho}((\widehat{Y}_t-\theta_t)+\frac{2-v}{2}(\widehat{p}^*_{hft}+\frac{1}{\rho}\widehat{e}_t-\widehat{p}_{hh,t}+\widehat{t}^*_{c,t})) \\ & -\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}((\widehat{Y}_t-\theta_t)-\frac{v}{2}(\widehat{p}^*_{hft}+\frac{1}{\rho}\widehat{e}_t-\widehat{p}_{hh,t}+\widehat{t}^*_{c,t})) \\ & -\delta\frac{2-v}{v}\varpi_{1,t}-\varpi_{4,t}\end{aligned} \tag{5.252}$$

Replace $\widehat{C}_t$ and simplify the equations:

$$\begin{aligned}
0 \quad = \quad & \frac{1}{2}\,(v\rho - v + 2)\,(\widehat{Y}_t - \theta_t) - \frac{\rho}{2}(\widehat{Y}_t - \theta_t) \\
& + \frac{1}{2\rho}\,(\rho - 1)\,(v\rho - v + 2)\,(\theta_t - \frac{2 - v}{2}\widehat{q}_t + \frac{2 - v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})) \\
& - \frac{1}{2}\,(\rho - 1)\,(\theta_t - \frac{2 - v}{2}\widehat{q}_t + \frac{2 - v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})) \\
& - \frac{1}{4\rho}\,(v - 2)\,(v\rho - v + 2)\,(\widehat{d}_t + \widehat{t}^*_{c,t}) \\
& + \frac{1}{4\rho}\,(v - 2)\,(\widehat{d}_t + \widehat{t}^*_{c,t}) \\
& + \frac{1}{4\rho}\,(\rho - 1)\,(v - 2)\,(v\rho - v + 2)\,\widehat{e}_t \\
& - \frac{1}{4}\,(\rho - 1)\,(v - 2)\,\widehat{e}_t \\
& + \frac{1}{2v}\,(v\rho - v + 2)\,\delta\varpi_{1,t} + \frac{1}{2}\varpi_{4,t}
\end{aligned} \tag{5.253}$$

Using equation (5.246), we have:

$$(\widehat{d}_t + \widehat{t}^*_{c,t}) = -\frac{2\rho\delta}{v}\varpi_{1,t} - \frac{2\rho}{(v\rho - v + 2)}\varpi_{4,t} \tag{5.254}$$

We can simplify the FOC of $(\widehat{Y}_t - \theta_t)$:

$$\begin{aligned}
& \frac{1}{2}(\widehat{Y}_t - \theta_t) \\
& + \frac{1}{2\rho}\,(\rho - 1)\,(\theta_t - \frac{2 - v}{2}\widehat{q}_t + \frac{2 - v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})) \\
& - \frac{v}{4\rho}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{1}{4\rho}\,(\rho - 1)\,(v - 2)\,\widehat{e}_t = 0
\end{aligned} \tag{5.255}$$

Similarly, we have

$$\begin{aligned}
& \frac{1}{2}(\widehat{Y}^*_t - \theta^*_t) \\
& + \frac{1}{2\rho}\,(\rho - 1)\,(\theta^*_t + \frac{2 - v}{2}\widehat{q}_t - \frac{2 - v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})) \\
& - \frac{v}{4\rho}(\widehat{d}^*_t + \widehat{t}_{c,t}) - \frac{1}{4\rho}\,(\rho - 1)\,(v - 2)\,\widehat{e}_t = 0
\end{aligned} \tag{5.256}$$

The difference of these two equations are

$$\begin{aligned} &\frac{1}{2}(\widehat{Y}_t-\widehat{Y}_t^*)-\frac{1}{2}(\theta_t-\theta_t^*) \\ &+\frac{(\rho-1)}{2\rho}(\theta_t-\theta_t^*)-\frac{(\rho-1)}{2\rho}(2-v)\widehat{q}_t+\frac{(\rho-1)}{2\rho}(2-v)(\widehat{t}_{c,t}^*-\widehat{t}_{c,t}) \\ &-\frac{v}{4\rho}(\widehat{d}_t+\widehat{t}_{c,t}^*)+\frac{v}{4\rho}(\widehat{d}_t^*+\widehat{t}_{c,t})+\frac{1}{2\rho}\left(\rho-1\right)\left(v-2\right)\widehat{e}_t=0 \end{aligned} \tag{5.257}$$

By replacing $\widehat{Y}_t-\widehat{Y}_t^*$ using equation (5.227):

$$\begin{aligned} &\frac{1}{2}(\frac{v}{2}\widehat{d}_t-\frac{v}{2}\widehat{d}_t^*+\widehat{q}_t+\frac{2-v}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*)+(v-1)(\frac{1}{\rho}-1)\widehat{e}_t) \\ &+\frac{-1}{2\rho}(\theta_t-\theta_t^*)-\frac{(\rho-1)}{2\rho}(2-v)\widehat{q}_t+\frac{(\rho-1)}{2\rho}(2-v)(\widehat{t}_{c,t}^*-\widehat{t}_{c,t}) \\ &-\frac{v}{4\rho}(\widehat{d}_t+\widehat{t}_{c,t}^*)+\frac{v}{4\rho}(\widehat{d}_t^*+\widehat{t}_{c,t})+\frac{1}{2\rho}\left(\rho-1\right)\left(v-2\right)\widehat{e}_t=0 \end{aligned} \tag{5.258}$$

By replacing $\widehat{e}_t$:

$$\begin{aligned} &\frac{1}{2}(\frac{v}{2}\widehat{d}_t-\frac{v}{2}\widehat{d}_t^*+\widehat{q}_t+\frac{2-v}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*)) \\ &-\frac{1}{2\rho}\left(\rho-1\right)(\frac{v}{2}\widehat{d}_t-\frac{v}{2}\widehat{d}_t^*-(1-v)\widehat{q}_t-\frac{2-v}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*)) \\ &+\frac{-1}{2\rho}(\theta_t-\theta_t^*)-\frac{(\rho-1)}{2\rho}(2-v)\widehat{q}_t+\frac{(\rho-1)}{2\rho}(2-v)(\widehat{t}_{c,t}^*-\widehat{t}_{c,t}) \\ &-\frac{v}{4\rho}(\widehat{d}_t+\widehat{t}_{c,t}^*)+\frac{v}{4\rho}(\widehat{d}_t^*+\widehat{t}_{c,t})=0 \end{aligned} \tag{5.259}$$

Further simplify:

$$\widehat{q}_t=(\theta_t-\theta_t^*)-(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*) \tag{5.260}$$

Using equation (5.222),(5.223) and (5.260), we find that

$$\begin{aligned} &\rho\widehat{Y}_t-\theta_t+\frac{2-v}{2}(\widehat{d}_t+\widehat{t}_{c,t}^*)+\frac{2-v}{2}(1-\rho)(\widehat{q}_t+\widehat{e}_t)+\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*) \\ =\;&\rho\widehat{Y}_t^*-\theta_t^*-\frac{v}{2}(\widehat{d}_t^*+\widehat{t}_{c,t})+\frac{(2-v)}{2}(1-\rho)(\widehat{q}_t^*-\widehat{e}_t)+\frac{(2-v)(\rho-1)}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*)+(\widehat{d}_t+\widehat{t}_{c,t}^*) \end{aligned} \tag{5.261}$$

Know that equation (??) and (??) implies:

$$\rho\widehat{Y}_t^*-\theta_t^*-\frac{v}{2}(\widehat{d}_t^*+\widehat{t}_{c,t})+\frac{(2-v)}{2}(1-\rho)(\widehat{q}_t^*-\widehat{e}_t)+\frac{(2-v)(\rho-1)}{2}(\widehat{t}_{c,t}-\widehat{t}_{c,t}^*)=0 \tag{5.262}$$

Therefore, using (5.221) and (5.231),we have:

$$\varpi_{2,t} - \varpi_{2,t-1} = 0 + \beta E_t(\varpi_{2,t+1} - \varpi_{2,t}) \tag{5.263}$$

Similarly, from equation (5.220),(5.246),(5.250), and FOCs (5.230), we have the following equations for $\varpi_{1,t}$ and $\varpi_{2,t}$:

$$\varpi_{1,t} - \varpi_{1,t-1} = \delta\left[-\frac{2\rho}{v}\delta\varpi_{1,t} - \frac{2\rho(\rho-1)\delta}{2-v+v\rho}\varpi_{2,t}\right] + \beta E_t(\varpi_{1,t} - \varpi_{1,t-1}) \tag{5.264}$$

Then, we must have $\varpi_{2,t} = \varpi_{1,t} = \varpi_{4,t} = 0$ and

$$\widehat{d}_t + \widehat{t}^*_{c,t} = 0 \tag{5.265}$$

Similarly, for the Foreign country, we have

$$\varpi^*_{2,t} = \varpi^*_{1,t} = \widehat{d}^*_t + \widehat{t}_{c,t} = 0 \tag{5.266}$$

which also implies $\pi_{hh,t} = 0$, $\pi^*_{ff,t} = 0, \pi^*_{hf,t} = 0$, $\pi_{fh,t} = 0$. Put these solution into expression of (5.229) and (5.236), we solve optimal output gap.

$$(\widehat{Y}_t - \theta_t) = -\frac{1}{2\rho}(\rho-1)(\theta_t+\theta^*_t) - \frac{1}{2\rho}(\rho-1)(v-1)\widehat{e}_t \tag{5.267}$$

$$(\widehat{Y}^*_t - \theta^*_t) = -\frac{1}{2\rho}(\rho-1)(\theta_t+\theta^*_t) + \frac{1}{2\rho}(\rho-1)(v-1)\widehat{e}_t \tag{5.268}$$

$$\begin{aligned} \widehat{C}_t &= \frac{(\theta_t+\theta^*_t)}{2\rho} + \frac{\widehat{e}_t}{2\rho} \\ \widehat{C}^*_t &= \frac{(\theta_t+\theta^*_t)}{2\rho} - \frac{\widehat{e}_t}{2\rho} \\ \widehat{e}_t &= (v-1)(\theta_t - \theta^*_t) \end{aligned} \tag{5.269}$$

Conclusion: The optimal policy under noncorperation game (with restriction) is identical to the ones under corperation game.

QED.

6 Three-players Nash game

In brief, the monetary authorities care domestic loss function and minimize the welfare loss given the constraints and the other players choices. The fiscal alliance minimizes the global welfare loss.

It's easy to verify that the following solution is the equilibrium of this three-player Nash game for both under $\rho > 1$ and $\rho = 1$.

6.1 The Game, $\rho = 1$

- The problems of Home and Foreign monetary authorities are identical the the Nash game $(P2)$ except that $\left\{\widehat{t}_{c,t}, \widehat{s}_{e,t}, \widehat{t}^*_{c,t}, \widehat{s}^*_{e,t}\right\}$ are not their choice variables. $\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}$ are shadow price for inflation dynamcis constraints and resource constraint at Home while $\varphi^*_{1,t}, \varphi^*_{2,t}, \varphi^*_{3,t}$ are shadow price for constraints in foreign country respectively.. First order conditions are:

$$\widehat{Y}_t - \theta_t \ : \ (\widehat{Y}_t - \theta_t) - \frac{(2-v)}{2}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\varphi_{1,t} - \varphi_{3,t} = 0 \tag{6.270}$$

$$\pi_{hh,t} \ : \ \frac{\lambda v}{2\delta}\pi_{hh,t} + \varphi_{1,t-1} - \varphi_{1,t} = 0 \tag{6.271}$$

$$\pi_{fh,t} \ : \ \frac{\lambda(2-v)}{2\delta}\pi_{fh,t} + \varphi_{2,t-1} - \varphi_{2,t} = 0 \tag{6.272}$$

$$\widehat{d}_t \ : \ \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t})\right) + \delta\frac{2-v}{2}\varphi_{1,t} + \frac{v}{2}\varphi_{3,t} = 0 \tag{6.273}$$

$$\widehat{q}_t \ : \ \varphi_{3,t} = 0 \tag{6.274}$$

$$\widehat{Y}^*_t - \theta^*_t \ : \ (\widehat{Y}^*_t - \theta^*_t) - \frac{(2-v)}{2}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\varphi^*_{1,t} + \varphi^*_{3,t} = 0 \tag{6.275}$$

$$\pi^*_{ff,t} \ : \ \frac{\lambda v}{2\delta}\pi^*_{ff,t} + \varphi^*_{1,t-1} - \varphi^*_{1,t} = 0 \tag{6.276}$$

$$\pi^*_{hf,t} \ : \ \frac{\lambda(2-v)}{2\delta}\pi^*_{hf,t} + \varphi^*_{2,t-1} - \varphi^*_{2,t} = 0 \tag{6.277}$$

$$\widehat{d}^*_t \ : \ \frac{v(2-v)}{4}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{v(2-v)}{4}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\right) + \delta\frac{2-v}{2}\varphi^*_{1,t} - \frac{v}{2}\varphi^*_{3,t} = 0 \tag{6.278}$$

$$\widehat{q}^*_t \ : \ \varphi^*_{3,t} = 0 \tag{6.279}$$

Rearrange the optimal choices (6.270)− (6.273) and (6.270)− (6.273) , we have:

$$\begin{aligned}(\widehat{Y}_t - \theta_t) &= -\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) \\ (\widehat{Y}^*_t - \theta^*_t) &= -\frac{(2-v)}{2}(\widehat{d}^*_t + \widehat{t}_{c,t})\end{aligned}$$

$$\begin{aligned}\varphi_{1,t} &= 0 \\ \varphi^*_{1,t} &= 0\end{aligned}$$

- The problem of global fiscal alliance: the fiscal authorities can coordinate. While fiscal coordinating, the authorities maximize the global welfare. The objective fucntion and constraints are presented in $(P1)$. $\xi_{5,t}$ is the lagrangian multiplier of corresponding constraints of market clearing condition. The FOC of global fiscal authority on $\widehat{s}^*_{e,t}$ and $\widehat{s}_{e,t}$:

$$\widehat{s}_{e,t} : \pi^*_{hf,t} = 0 \tag{6.280}$$

$$\widehat{s}^*_{e,t} : \pi_{fh,t} = 0 \tag{6.281}$$

It's easy to see when fiscal authority commit to a subsidy rule at time 0, the first order conditions implies that optimal inflation rates are zero. The FOC on $\widehat{t}_{c,t}$ and $\widehat{t}^*_{c,t}$ are:

$$\widehat{t}_{c,t} \; : \; \frac{v(2-v)}{4}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{2-v}{2}\pi^*_{ff,t} + \frac{2-v}{2}\pi_{fh,t} + \frac{2-v}{2}\xi_{5,t} = 0 \tag{6.282}$$

$$\widehat{t}^*_{c,t} \; : \; \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}\pi_{hh,t} + \frac{2-v}{2}\pi^*_{hf,t} - \frac{2-v}{2}\xi_{5,t} = 0 \tag{6.283}$$

In equilibrium, $\pi^*_{hf,t} = \pi_{fh,t} = 0$ implies

$$\begin{aligned}\widehat{d}_t + \widehat{s}_{e,t} &= 0 \\ \widehat{d}^*_t + \widehat{s}^*_{e,t} &= 0\end{aligned}$$

Through the FOC of monetary authority, we know that $\pi_{hh,t} = \pi^*_{ff,t} = 0$. Using the market clearing condition and the FOCs on $\widehat{t}_{c,t}$ and $\widehat{t}^*_{c,t}$, we know that the optimal output gaps are:

$$(\widehat{Y}_t - \theta_t) = 0 \tag{6.284}$$

$$(\widehat{Y}^*_t - \theta^*_t) = 0 \tag{6.285}$$

Given that

$$\begin{aligned}
\pi^*_{hf,t+1} &= -\delta \sum_{s=0}^{\infty} \beta^s E_t \left[\widehat{d}_{t+1+s} + \widehat{s}_{e,t+1+s} \right] = 0 \\
\pi_{fh,t+1} &= -\delta \sum_{s=0}^{\infty} \beta^s E_t \left[\widehat{d}^*_{t+1+s} + \widehat{s}^*_{e,t+1+s} \right] = 0
\end{aligned}$$

$$\begin{aligned}
\pi^*_{hf,t} &= 0 \Longrightarrow \widehat{t}^*_{c,t} = \widehat{s}_{e,t} \\
\pi_{fh,t} &= 0 \Longrightarrow \widehat{t}_{c,t} = \widehat{s}^*_{e,t}
\end{aligned}$$

To this end, this two-step game replicates our previous result. This result can be extended to the general case $\rho > 1$.

6.2 The Game, $\rho > 1$

- The problem of Home monetary authority

Home loss function:

$$(P5): L_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\ +\frac{v(2-v)}{8}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})^2 \\ +\frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] \\ +\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi^{*2}_{ff,t} + \frac{(2-v)}{2}\pi^{*2}_{hf,t}\right] \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)^2 - \frac{1-\rho}{2}\widehat{C}^2_t \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}^*_t + \widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1}{\rho}(1-\rho)^2(\widehat{C}^2_t - \widehat{C}^{*2}_t) \\ +\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) - (\widehat{t}^*_{c,t} - \widehat{s}_{e,t})\right) \\ +\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ -\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})\right)^2 \end{array} \right\} \quad (6.286)$$

The **Home** monetary authority minimizes loss function by choosing $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{q}_t, \widehat{e}_t\right\}$ given $\{\widehat{Y}^*_t - \theta^*_t, \pi^*_{hf,t}, \pi^*_{ff,t}, \widehat{d}^*_t, \widehat{t}_{c,t}, \widehat{t}^*_{c,t}, \widehat{s}_{e,t}, \widehat{s}^*_{e,t}\}$, subject to following constraint, shadow prices are $\{\varpi_{1,t}, \varpi_{2,t}, \varpi_{3,t}, , \varpi_{4,t}\}$:

$$\pi_{hh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \end{array} \right] + \beta E_t \pi_{hh,t+1} \quad (6.287)$$

$$\pi_{fh,t} = \delta \left[\begin{array}{c} \rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) \\ +\frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t} - \widehat{s}^*_{e,t} \end{array} \right] + \beta E_t \pi_{fh,t+1} \quad (6.288)$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \quad (6.289)$$

$$\widehat{e}_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t - (1-v)\widehat{q}_t - \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \quad (6.290)$$

- The FOCs of Home monetary authority:

We define

$$
\begin{aligned}
M_1 &= (\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t}) \\
M_2 &= (\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t}) \\
M_3 &= (\widehat{Y}^*_t - \theta^*_t) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t}) \\
M_4 &= (\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}^*_{hft} + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}^*_{c,t})
\end{aligned}
$$

The first order conditions of $\widehat{Y}_t - \theta_t$:

$$
\begin{gathered}
(\widehat{Y}_t - \theta_t) - (1-\rho)\widehat{C}_t + \frac{v(2-v)}{8}\frac{1-\rho}{\rho}2M_1 \\
-\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}2M_4 + \delta\rho\varpi_{1,t} - \varpi_{3,t} \\
-\frac{v(2-v)(1-\rho)}{4}\frac{\rho - 1 + \frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) \\
+\frac{v(2-v)}{4}\frac{\rho-1}{\rho}(1-\rho)^2\widehat{C}_t = 0
\end{gathered} \tag{6.291}
$$

The first order conditions of $\pi_{hh,t}$ and $\pi_{fh,t}$:

$$
\pi_{hh,t} : \frac{\lambda(1+\frac{\alpha}{\rho})v}{4\delta}\pi_{hh,t} + \varpi_{1,t-1} - \varpi_{1,t} = 0 \tag{6.292}
$$

$$
\pi_{fh,t} : \frac{\lambda(1+\frac{\alpha}{\rho})(2-v)}{4\delta}\pi_{fh,t} + \varpi_{2,t-1} - \varpi_{2,t} = 0 \tag{6.293}
$$

The first order conditions of $\widehat{d}_t$:

$$
\begin{gathered}
\frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t) + \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_1 \\
+\frac{v(2-v)}{8}\frac{\rho - 1 + \frac{2}{v}}{\rho}vM_4 + \delta\frac{2-v}{2}\varpi_{1,t} + \frac{v}{2}\varpi_{3,t} + \frac{v}{2}\varpi_{4,t} = 0
\end{gathered} \tag{6.294}
$$

The first order conditions of $\widehat{e}_t$:

$$
\begin{aligned}
0 = {} & \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{c,t} + \frac{1}{\rho}\widehat{e}_t - \widehat{e}_t)\frac{1-\rho}{\rho} \\
& -(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}\widehat{C}_t \\
& -\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2} \\
& -\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}^*_t + \widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2} \\
& +\frac{v(2-v)}{4}\frac{\rho-1}{\rho}(1-\rho)^2\frac{(2-v)}{2}\frac{1-\rho}{\rho}\left(-\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}_t\right) \\
& +\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_1 + \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_2 \\
& +\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_3 + \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_4 \\
& +\frac{(2-v)(1-\rho)}{2}\delta\varpi_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi_{2,t} + \varpi_{3,t}(\frac{1}{\rho}-1)(v-1) - \varpi_{4,t}
\end{aligned}
\tag{6.295}
$$

First order condition of $\widehat{q}_t$:

$$
\begin{aligned}
0 \quad = \quad & (1-\rho)\frac{(2-v)}{2}\frac{1-\rho}{\rho}\widehat{C}_t \\
& +\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2} \\
& +\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}^*_t + \widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2} \\
& -\frac{v(2-v)}{4}\frac{\rho-1}{\rho}(1-\rho)^2\frac{(2-v)}{2}\frac{1-\rho}{\rho}\left(-\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}_t\right) \\
& +\frac{(2-v)(1-\rho)}{2}\frac{1-\rho}{\rho}\delta\varpi_{1,t} - \frac{(2-v)(1-\rho)}{2}\frac{1-\rho}{\rho}\delta\varpi_{2,t} \\
& +\frac{1-\rho}{\rho}\varpi_{3,t} - \frac{1-\rho}{\rho}(1-v)\varpi_{4,t}
\end{aligned}
\tag{6.296}
$$

- The problem of Foreign monetary authority:

Foreign loss function:

$$(P5): L_{h,0}^* = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t^* - \theta_t^*)^2 \\ +\frac{v(2-v)}{8}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})^2 \\ +\frac{\lambda(1-\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{hh,t}^2 + \frac{(2-v)}{2}\pi_{fh,t}^2\right] + \frac{\lambda(1+\frac{\alpha}{\rho})}{4\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right)^2 - \frac{1-\rho}{2}\widehat{C}_t^{*2} \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t^* + \widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1}{\rho}(1-\rho)^2(\widehat{C}_t^2 - \widehat{C}_t^{*2}) \\ -\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{t}_{c,t} - \widehat{s}_{e,t}^*) - (\widehat{t}_{c,t}^* - \widehat{s}_{e,t})\right) \\ -\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t - \theta_t) + \frac{2-v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2 \\ -\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{1-\rho}{\rho}\left((\widehat{Y}_t^* - \theta_t^*) + \frac{2-v}{2}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{ff,t}^* + \widehat{t}_{c,t})\right)^2 \\ +\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}\left((\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{p}_{hft}^* + \frac{1}{\rho}\widehat{e}_t - \widehat{p}_{hh,t} + \widehat{t}_{c,t}^*)\right)^2 \end{array} \right\} \tag{6.297}$$

The foreign monetary authority minimize loss function by choosing $\left\{\widehat{Y}_t^* - \theta_t^*, \pi_{hf,t}^*, \pi_{ff,t}^*, \widehat{d}_t^*, \widehat{q}_t^*, \widehat{e}_t^*\right\}$ given $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{c,t}, \widehat{t}_{c,t}^*\right\}$, subject to following constraint, shadow prices are $\{\varpi_{1,t}^*, \varpi_{2,t}^*, \varpi_{3,t}^*, \varpi_{4,t}^*\}$:

$$\pi_{hf,t}^* = \delta\left[\begin{array}{c} \rho\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^*) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) \\ -\frac{(2-v)}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}_{c,t}^*) + \widehat{t}_{c,t}^* - \widehat{s}_{e,t} \end{array}\right] + \beta E_t \pi_{hf,t+1}^* \tag{6.298}$$

$$\pi_{ff,t}^* = \delta\left[\begin{array}{c} \rho\widehat{Y}_t^* - \theta_t^* + \frac{2-v}{2}(\widehat{d}_t^* + \widehat{t}_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t^* - \widehat{e}_t) \\ +\frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t}^* - \widehat{t}_t^c) \end{array}\right] + \beta E_t \pi_{ff,t+1}^* \tag{6.299}$$

$$\widehat{Y}_t - \widehat{Y}_t^* = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}_t^* + \widehat{q}_t + \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}_{c,t}^*) + (v-1)(\frac{1}{\rho} - 1)\widehat{e}_t \tag{6.300}$$

$$\widehat{e}_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}_t^* - (1-v)\widehat{q}_t - \frac{2-v}{2}(\widehat{t}_{c,t} - \widehat{t}_{c,t}^*) \tag{6.301}$$

- The FOC of Foreign monetary authority:

The first order condition of $\widehat{Y}_t^* - \theta_t^*$:

$$\widehat{Y}_t^* - \theta_t^* : (\widehat{Y}_t^* - \theta_t^*) + \frac{v(2-v)}{8}\frac{1-\rho}{\rho}2M_3 \tag{6.302}$$
$$-\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}2M_2 + \delta\rho\varpi_{2,t}^* + \varpi_{3,t}^* - (1-\rho)\widehat{C}_t^*$$
$$-\frac{v(2-v)(1-\rho)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t^* + \widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right)$$
$$+\frac{v(2-v)}{4}\frac{\rho-1}{\rho}(1-\rho)^2\widehat{C}_t^* = 0$$

The first order condition of $\pi_{hf,t}^*$ and $\pi_{ff,t}^*$:

$$\pi_{hf,t}^* : \frac{\lambda(1+\frac{\alpha}{\rho})(2-v)}{4\delta}\pi_{hf,t}^* + \varpi_{1,t-1}^* - \varpi_{1,t}^* = 0 \tag{6.303}$$
$$\pi_{ff,t}^* : \frac{\lambda(1+\frac{\alpha}{\rho})v}{4\delta}\pi_{ff,t}^* + \varpi_{2,t-1}^* - \varpi_{2,t}^* = 0 \tag{6.304}$$

The first order condition of $\widehat{d}_t^*$:

$$\widehat{d}_t^* : \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{c,t} - \frac{1}{\rho}\widehat{e}_t + \widehat{e}_t) + \frac{v(2-v)^2}{8}\frac{1-\rho}{\rho}M_3 \tag{6.305}$$
$$+\frac{v(2-v)}{8}\frac{\rho-1+\frac{2}{v}}{\rho}vM_2 + \delta\frac{2-v}{2}\varpi_{2,t}^* - \frac{v}{2}\varpi_{3,t}^* - \frac{v}{2}\varpi_{4,t}^* = 0$$

The first order conditions of $\widehat{e}_t$:

$$0 = -\frac{v(2-v)}{4}(\widehat{p}_{fht} - \frac{1}{\rho}\widehat{e}_t - \widehat{p}^*_{ff,t} + \widehat{t}_{c,t})\frac{1-\rho}{\rho} \tag{6.306}$$
$$+\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}$$
$$+\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}^*_t + \widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}$$
$$+(1-\rho)\frac{1-\rho}{\rho}\frac{(2-v)}{2}\widehat{C}^*_t$$
$$+\frac{v(2-v)}{4}\frac{1-\rho}{\rho}(1-\rho)^2\frac{(2-v)}{2}\frac{1-\rho}{\rho}\left(\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}^*_t\right)$$
$$-\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_1 - \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_2$$
$$-\frac{v(2-v)}{4}\frac{1-\rho}{\rho}\frac{(2-v)}{2}\frac{1-\rho}{\rho}M_3 - \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\frac{v}{2}\frac{1-\rho}{\rho}M_4$$
$$+\frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{2,t} + \varpi^*_{3,t}(\frac{1}{\rho}-1)(v-1) - \varpi^*_{4,t}$$

First order condition of $\widehat{q}^*_t$:

$$\begin{aligned} 0 \;=\; & (1-\rho)\frac{(2-v)}{2}\widehat{C}^*_t \\ & -\frac{v(2-v)}{4}\frac{\rho-1}{\rho}(1-\rho)^2\frac{(2-v)}{2}\left(\frac{1}{\rho}\widehat{e}_t + 2\widehat{C}^*_t\right) \\ & +\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}_t + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)(1-\rho)\frac{(2-v)}{2} \\ & +\frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}\left((1-\rho)\widehat{C}^*_t + \widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)(1-\rho)\frac{(2-v)}{2} \\ & -\frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{1,t} + \frac{(2-v)(1-\rho)}{2}\delta\varpi^*_{2,t} - \varpi^*_{3,t} + (1-v)\varpi^*_{4,t} \end{aligned} \tag{6.307}$$

- The problem of global fiscal alliance, the problem is characterized in $(P3)$.Optimal choices are $\{\widehat{t}_{c,t}, \widehat{t}^*_{c,t}, \widehat{s}^*_{e,t}, \widehat{s}_{e,t}\}$. The constraints are (4.191)-(4.196). $\xi_{5,t}$ and $\xi_{6,t}$ are the lagrangian multipliers of corresponding constraints of market clearing condition and real exchange rate. The FOC of global fiscal authority on $\widehat{s}^*_{e,t}$ and $\widehat{s}_{e,t}$:

$$\widehat{s}_{e,t} : \pi^*_{hf,t} = 0 \tag{6.308}$$

$$\widehat{s}^*_{e,t} : \pi_{fh,t} = 0 \tag{6.309}$$

It's easy to see when fiscal authority commit to a subsidy rule at time 0, the first order conditions implies that optimal inflation rates are zero.

6.3 Proof

We denote M_1, M_2, M_3, M_4 as in the previous section. Since $\pi^*_{hf,t}$ and $\pi_{fh,t}$ are zero in the Nash equilibrium, the FOCs equations (6.293) and (6.303) imply $\varpi_{2,t} = \varpi^*_{1,t} = 0$.

- Combine FOC of $\widehat{d}_t$ and $\widehat{Y}_t - \theta_t$ and simplify the equations, we have:

$$\begin{aligned} &\frac{v}{2\rho}\begin{bmatrix} \rho(\widehat{Y}_t - \theta_t) \\ +\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) \\ +(\rho-1)\left(\theta_t - \frac{2-v}{2}(\widehat{q}_t + \widehat{e}_t) + \frac{2-v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})\right) \end{bmatrix} \\ &+\delta\varpi_{1,t} + \frac{v}{(2-v+\rho v)}\varpi_{4,t} - \frac{v(2-v)(1-\rho)}{4\rho}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) = 0 \end{aligned} \tag{6.310}$$

Similarly, we combine FOC of $\widehat{d}^*_t$ and $\widehat{Y}^*_t - \theta^*_t$, we have:

$$\begin{aligned} &\begin{bmatrix} \rho(\widehat{Y}^*_t - \theta^*_t) \\ +\frac{(2-v)}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) \\ +(\rho-1)\left(\theta^*_t + \frac{2-v}{2}(\widehat{q}_t + \widehat{e}_t) - \frac{2-v}{2}(\widehat{t}^*_{c,t} - \widehat{t}_{c,t})\right) \end{bmatrix} \\ = \quad & -\frac{2\rho}{v}\delta\varpi^*_{2,t} + \frac{2\rho}{2-v+v\rho}\varpi_{4,t} + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) \end{aligned}$$

This is the first key equation for our solution. We can express:

$$\pi_{hh,t} = \delta\left[-\frac{2\rho}{v}\delta\varpi_{1,t} - \frac{2\rho}{2-v+v\rho}\varpi_{4,t} + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)\right] + \beta E_t \pi_{hh,t+1} \tag{6.311}$$

$$\pi^*_{ff,t} = \delta\left[-\frac{2\rho}{v}\delta\varpi^*_{1,t} + \frac{2\rho}{2-v+v\rho}\varpi_{4,t} + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right)\right] + \beta E_t \pi^*_{ff,t+1} \tag{6.312}$$

- By replacing $\varpi_{3,t}$ in the FOC of $\widehat{q}_t$ using the FOC of $\widehat{d}_t$ and simplify the equation using (6.310):

$$\begin{aligned}\frac{1}{4\rho}(\rho-1)(v-2)^2(\widehat{d}_t + \widehat{t}^*_{c,t}) &= \frac{1}{8\rho}(\rho-1)^2(v-2)^3(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) \\ &+ \frac{v(2-v)}{4}\frac{\rho-1+\frac{2}{v}}{\rho}(1-\rho)\frac{(2-v)}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) \\ &- \frac{1}{2v}\delta(\rho-1)(v-2)^2\delta\varpi_{1,t} - \frac{(2-v)(1-\rho)}{2}\delta\varpi_{2,t} \\ &+ \rho\frac{v-2}{v\rho-v+2}\varpi_{4,t}\end{aligned} \tag{6.313}$$

Further simplify:

$$\begin{aligned}(\widehat{d}_t + \widehat{t}^*_{c,t}) &= \frac{(\rho-1)(v-2)}{2}(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) - \frac{v\rho-v+2}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) \\ &- \frac{2\rho}{v}\delta\varpi_{1,t} + \frac{2\rho\delta}{(2-v)}\varpi_{2,t} + \frac{4\rho^2}{(2-v+\rho v)(v-2)(\rho-1)}\varpi_{4,t}\end{aligned} \tag{6.314}$$

Similarly,

$$\begin{aligned}(\widehat{d}^*_t + \widehat{t}_{c,t}) &= \frac{(\rho-1)(v-2)}{2}(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}) - \frac{v\rho-v+2}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) \\ &- \frac{2\rho}{v}\delta\varpi^*_{2,t} + \frac{2\rho\delta}{(2-v)}\varpi^*_{1,t} - \frac{4\rho^2}{(2-v+\rho v)(v-2)(\rho-1)}\varpi^*_{4,t}\end{aligned} \tag{6.315}$$

The above is the second key equation of the proof.

- We further simplify the FOC of $\widehat{Y}_t - \theta_t$ in equation (6.310) and corresponding equation for the foreign country.

$$\begin{aligned}
&\rho(\widehat{Y}_t - \widehat{Y}_t^*) - \theta_t + \theta_t^* \\
&-\frac{(\rho-1)(v-2)^2}{4}(\widehat{t}_{c,t} - \widehat{s}_{e,t}^*) - \frac{v\rho - v + 2}{2}\frac{(2-v)}{2}\left(\widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right) \\
&+\rho\delta\varpi_{1,t} + \rho\delta\varpi_{2,t} - \frac{2\rho}{(2-v+\rho v)(\rho-1)}\varpi_{4,t} \\
&+\frac{(\rho-1)(v-2)^2}{4}(\widehat{t}_{c,t}^* - \widehat{s}_{e,t}) + \frac{v\rho - v + 2}{2}\frac{(2-v)}{2}\left(\widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right) \\
&-\rho\delta\varpi_{1,t}^* - \rho\delta\varpi_{2,t}^* - \frac{2\rho}{(2-v+\rho v)(\rho-1)}\varpi_{4,t}^* \\
&+(1-\rho)(2-v)(\widehat{q}_t + \widehat{e}_t - \widehat{t}_{c,t}^* + \widehat{t}_{c,t}) \\
&-\frac{(2-v)(1-\rho)}{2}\left(\widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right) + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right) = 0
\end{aligned} \tag{6.316}$$

Use the resource constraint and the defition of real exchange rate, we have:

$$\begin{aligned}
&(\frac{v}{2}(\widehat{d}_t + \widehat{t}_{c,t}^*) - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{c,t})) + \widehat{q}_t + (\widehat{t}_{c,t} - \widehat{t}_{c,t}^*) - \theta_t + \theta_t^* \\
&+\frac{2-v}{2}(v\rho - v + 1)\left(\widehat{t}_{c,t} - \widehat{s}_{e,t}^*\right) - \frac{2-v}{2}(v\rho - v + 1)\left(\widehat{t}_{c,t}^* - \widehat{s}_{e,t}\right) \\
&+\rho\delta\varpi_{1,t} + \rho\delta\varpi_{2,t} - \frac{2\rho}{(2-v+\rho v)(\rho-1)}\varpi_{4,t} \\
&-\rho\delta\varpi_{1,t}^* - \rho\delta\varpi_{2,t}^* - \frac{2\rho}{(2-v+\rho v)(\rho-1)}\varpi_{4,t}^* = 0
\end{aligned} \tag{6.317}$$

Using the expression of $(\widehat{d}_t + \widehat{t}_{c,t}^*)$ and $(\widehat{d}_t^* + \widehat{t}_{c,t})$, we have

$$\begin{aligned}
&\frac{(v\rho - v + 2)}{2}(\widehat{t}_{c,t} - \widehat{s}_{e,t}^*) - \frac{(v\rho - v + 2)}{2}(\widehat{t}_{c,t}^* - \widehat{s}_{e,t}) \\
&+\frac{2\rho\delta}{(2-v)}\varpi_{2,t} + \frac{2\rho}{(v-2)(\rho-1)}\varpi_{4,t} - \frac{2\rho\delta}{(2-v)}\varpi_{1,t}^* + \frac{2\rho}{(v-2)(\rho-1)}\varpi_{4,t}^* \\
&+\widehat{q}_t + (\widehat{t}_{c,t} - \widehat{t}_{c,t}^*) - \theta_t + \theta_t^* = 0
\end{aligned} \tag{6.318}$$

- By using output gap in the NKPC of foreign country, we have:

$$
\begin{aligned}
& \rho\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{2-v}{2}(1-\rho)(\widehat{q}_t + \widehat{e}_t) + \frac{(2-v)(1-\rho)}{2}(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) \\
& - \left[\rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) + \frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right] \\
= \; & \rho\frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) - \rho\frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \rho\left(\widehat{q}_t + (\widehat{t}_{c,t} - \widehat{t}^*_{c,t})\right) \\
& + (1-\rho)\left[\frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) - (1-v)(\widehat{q}_t + (\widehat{t}_{c,t} - \widehat{t}^*_{c,t}))\right] \\
& -\theta_t + \theta^*_t + (2-v)(1-\rho)(\widehat{q}_t + \widehat{t}_{c,t} - \widehat{t}^*_{c,t}) - \widehat{t}_{c,t} + \widehat{s}^*_{e,t} - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{c,t}) + \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + (\widehat{d}_t + \widehat{t}^*_{c,t}) \\
= \; & \widehat{q}_t - \theta_t + \theta^*_t + \widehat{t}_{c,t} - \widehat{t}^*_{c,t} + (\widehat{d}_t + \widehat{t}^*_{c,t}) - \widehat{t}_{c,t} + \widehat{s}^*_{e,t}
\end{aligned}
$$

In other word, we can represent:

$$
\begin{aligned}
& \rho\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{c,t}) + \frac{(2-v)}{2}(1-\rho)(\widehat{q}^*_t - \widehat{e}_t) + \frac{2-v}{2}(\rho-1)(\widehat{t}_{c,t} - \widehat{t}^*_{c,t}) + \widehat{t}_{c,t} - \widehat{s}^*_{e,t} \\
= \; & -\frac{2\rho\delta}{v}\varpi_{1,t} - \frac{2\rho}{(2-v+\rho v)}\varpi_{4,t} + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) - (\widehat{d}_t + \widehat{t}^*_{c,t}) \\
& - \left[\widehat{q}_t - \theta_t + \theta^*_t + \widehat{t}_{c,t} - \widehat{t}^*_{c,t}\right] \\
= \; & -\frac{2\rho}{(v-2)(\rho-1)}\varpi_{4,t} - \left[\widehat{q}_t - \theta_t + \theta^*_t + \widehat{t}_{c,t} - \widehat{t}^*_{c,t}\right] \\
& + \frac{v\rho - v + 2}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) - \frac{2\rho\delta}{(2-v)}\varpi_{2,t} \\
= \; & -\frac{2\rho}{(v-2)(\rho-1)}\varpi^*_{4,t} + \frac{v\rho - v + 2}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) - \frac{2\rho\delta}{(2-v)}\varpi^*_{1,t}
\end{aligned}
$$

Since we have

$$\varpi_{2,t} = \varpi^*_{1,t} = 0$$

Which means

$$-\frac{2\rho}{(v-2)(\rho-1)}\varpi^*_{4,t} + \frac{v\rho - v + 2}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right) = 0$$

Similarly, we have

$$
\begin{aligned}
(\widehat{d}_t + \widehat{t}^*_{c,t}) &= \frac{(\rho-1)(v-2)}{2}(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}) \\
&\quad -\frac{v\rho - v + 2}{2}\left(\widehat{t}^*_{c,t} - \widehat{s}_{e,t}\right) - \frac{2\rho}{v}\delta\varpi_{1,t} \\
&\quad +\frac{4\rho^2}{(2-v+\rho v)(v-2)(\rho-1)}\varpi_{4,t}
\end{aligned} \tag{6.319}
$$

And

$$
(\widehat{d}_t + \widehat{t}^*_{c,t}) - (\widehat{t}^*_{c,t} - \widehat{s}_{e,t}) = -\frac{2\rho}{v}\delta\varpi_{1,t} - \frac{2\rho}{2-v+v\rho}\varpi_{4,t} + \frac{(2-v)(1-\rho)}{2}\left(\widehat{t}_{c,t} - \widehat{s}^*_{e,t}\right)
$$

Simplifying this equation, we have

$$
\begin{aligned}
\widehat{t}^*_{c,t} - \widehat{s}_{e,t} &= \Phi_1\varpi_{4,t} \\
\widehat{t}_{c,t} - \widehat{s}^*_{e,t} &= \Phi_2\varpi^*_{4,t}
\end{aligned}
$$

Φ_1 and Φ_2 are some constant, and we have $\varpi_{4,t} = \varpi^*_{4,t} = \widehat{t}^*_{c,t} - \widehat{s}_{e,t} = \widehat{t}_{c,t} - \widehat{s}^*_{e,t} = 0$

And thereby, using (6.311) and (6.312), we have

$$
\varpi_{1,t} = \varpi^*_{2,t} = 0
$$

$$
\begin{aligned}
\widehat{d}_t + \widehat{t}^*_{c,t} &= 0 \\
\widehat{d}^*_t + \widehat{t}_{c,t} &= 0
\end{aligned}
$$

It's easy to verify that the output gaps are still the same as in the coorperation game.

7 Welfare comparison with Engel (2011)

It's necessary to point out that the welfare loss function in Engel (2011) is measured by deviation from flexible price allocation. Below, we list the optimal policy in Engel (2011) and we show that this welfare loss (relative to flexible price) is positive given the optimal policy. We then present the flexible price allocations in Engel (2011). Finally, we show that our policy and allocations are identical to the flexible price allocation in Engel (2011). Thereby, we prove that our policy improves the global welfare.

Define $x_t^R = \frac{x_t - x_t^*}{2}$ and $x_t^W = \frac{x_t + x_t^*}{2}$. In his appendix, Engel defines the flexible price equilibrium, and calls this equilibrium as the efficient equilibrium. The welfare loss Ψ is calculated as the deviation from this equilibrium.

$$
\begin{aligned}
\Psi &= (\frac{\rho}{D})(\widetilde{y}^R)^2 + (\rho)(\widetilde{y}^W)^2 + \frac{v(2-v)}{4} m_t^2 \\
&\quad + \frac{\lambda}{\delta}\left[(\pi_t^R)^2 + (\pi_t^W)^2 + \frac{v(2-v)}{4}(s_t - s_{t-1})^2\right]
\end{aligned}
$$

$D = \rho v(2-v) + (v-1)^2$. The cooperation game (without markup shock) in Engel (2011) have the following solution:

$$
\begin{aligned}
\widetilde{y}^R &= \frac{(2-v)v}{2}\widetilde{s}_t, \widetilde{y}^W = 0 \\
m_t &= -(v-1)\widetilde{s}_t \\
\widetilde{s}_t &= s_t - (\theta_t - \theta_t^*) \\
s_t - s_{t-1} &= -\delta\widetilde{s}_t + \beta E_t(s_{t+1} - s_t) \\
\pi_t^R &= \pi_t^W = 0
\end{aligned}
$$

Under any solution this welfare cost Ψ is positive. It's easy to verify that our optimal allocations (5.265)-(5.269) are exactly identical to the efficient (flexible price) allocations in Engel (2011). Thereby, we prove that the welfare loss in our analysis is smaller than the welfare loss in Engel (2011).

8 Consider Tax instruments in Adao, Correia and Teles (2009)

In Adao, Correia and Teles (2009), they consider consumption tax on both domestic and imported good, as well as labor income tax. We discuss the role of this tax combination in our standard two country model with complete market across country. Model specification is similar to our basic dynamic model. Here we only highlight the equations which is changed when considering various tax instruments. Our conclusion is that combination of consumption tax and labor tax cannot achieve flexible price allocation in standard setting.

8.1 Model

Household budget constraint:

$$\begin{aligned} & (1+t_{h,t})P_{hht}C_{h,t} + (1+t_{f,t})P_{fht}C_{f,t} + M_{t+1} + B_{ht+1} + \sum_{\zeta^{t+1}\in Z_{t+1}} B(\zeta^{t+1}|\zeta^t)D(\zeta^{t+1}) \\ = \; & (1-\tau_t)W_tL_t + R_{t-1}B_{ht} + \Pi_t + M_t + T_t + D(\zeta^t) \end{aligned}$$

First order conditions

$$(1-\tau_t)W_t\frac{C_t^{-\rho}}{P_t} = \eta \tag{8.320}$$

Demand of home and foreign good

$$C_{h,t} = \frac{v}{2}\frac{P_tC_t}{P_{hht}(1+t_{h,t})} \tag{8.321}$$

$$C_{f,t} = (1-\frac{v}{2})\frac{P_tC_t}{P_{fht}(1+t_{f,t})} \tag{8.322}$$

And firms optimal pricing equations are different from the case when export subsidies are introduced. But these equations are standard in literature and omitted here.

market clearing condition

$$Y_t = \frac{v}{2}\frac{P_tC_t}{P_{hht}(1+t_{h,t})}\Delta_{hh,t} + (1-\frac{v}{2})\frac{P_t^*C_t^*}{P_{hft}^*(1+t_{h,t}^*)}\Delta_{hf,t}^* \tag{8.323}$$

8.2 loss function and coordination game

Following the same procedure as in our benchmark model ($\rho = 1$), we get the global loss function:

$$(P6) : L^w = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t^* - \theta_t^*)^2 + \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\ +\frac{v(2-v)}{8}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*)^2 + \frac{v(2-v)}{8}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t})^2 \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{hh,t}^{2} + \frac{(2-v)}{2}\pi_{fh,t}^{2}\right] \end{array} \right\} \tag{8.324}$$

NKPCs and resource constraint:

$$\pi_{hh,t} = \delta\left[\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t}) + \widehat{t}_{h,t} + \widehat{\tau}_t\right] + \beta E_t \pi_{hh,t+1} \tag{8.325}$$

$$\pi_{hf,t}^* = \delta\left[\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t}) + \widehat{t}_{h,t}^* + \widehat{\tau}_t\right] + \beta E_t \pi_{hf,t+1}^* \tag{8.326}$$

$$\pi_{ff,t}^* = \delta\left[\widehat{Y}_t^* - \theta_t^* + \frac{2-v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{t}_{f,t}^* + \widehat{\tau}_t^*\right] + \beta E_t \pi_{ff,t+1}^* \tag{8.327}$$

$$\pi_{fh,t} = \delta\left[\widehat{Y}_t^* - \theta_t^* - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{t}_{f,t} + \widehat{\tau}_t^*\right] + \beta E_t \pi_{fh,t+1} \tag{8.328}$$

$$\widehat{Y}_t - \widehat{Y}_t^* = -\frac{v}{2}\widehat{p}_{hh,t} - \frac{2-v}{2}(\widehat{p}_{hf,t}^*) + \frac{v-1}{\rho}\widehat{e}_t + \frac{v}{2}\widehat{p}_{ff,t}^* + \frac{2-v}{2}\widehat{p}_{fh,t} - \frac{v}{2}(\widehat{t}_{h,t} - \widehat{t}_{f,t}^*) + \frac{2-v}{2}(\widehat{t}_{f,t} - \widehat{t}_{h,t}^*) \tag{8.329}$$

- Let $\phi_{1,t}^A, \phi_{2,t}^A, \phi_{3,t}^A, \phi_{4,t}^A, \phi_{5,t}^A$ to be lagrangian multipliers for equations. First order conditions are

$$\begin{aligned}
\widehat{Y}_t-\theta_t &: \quad \widehat{Y}_t-\theta_t+\delta(\phi^A_{1,t}+\phi^A_{2,t})-\phi^A_{5,t}=0 && (8.330)\\
\widehat{Y}^*_t-\theta^*_t &: \quad \widehat{Y}^*_t-\theta^*_t+\delta(\phi^A_{3,t}+\phi^A_{4,t})+\phi^A_{5,t}=0 && (8.331)\\
\pi_{hh,t} &: \quad \frac{v\lambda}{2\delta}\pi_{hh,t}-\phi^A_{1,t}+\phi^A_{1,t-1}=0 && (8.332)\\
\pi^*_{hf,t} &: \quad \frac{(2-v)\lambda}{2\delta}\pi^*_{hf,t}-\phi^A_{2,t}+\phi^A_{2,t-1}=0 && (8.333)\\
\pi^*_{ff,t} &: \quad \frac{v\lambda}{2\delta}\pi^*_{ff,t}-\phi^A_{3,t}+\phi^A_{3,t-1}=0 && (8.334)\\
\pi_{fh,t} &: \quad \frac{(2-v)\lambda}{2\delta}\pi_{fh,t}-\phi^A_{4,t}+\phi^A_{4,t-1}=0 && (8.335)\\
\widehat{d}^*_t &: \quad \frac{v(2-v)}{4}(\widehat{d}^*_t+\widehat{t}_{f,t}-\widehat{t}^*_{f,t})+\delta(\frac{2-v}{2}\phi^A_{3,t}-\frac{v}{2}\phi^A_{4,t})-\frac{v}{2}\phi^A_{5,t}=0 && (8.336)\\
\widehat{d}_t &: \quad \frac{v(2-v)}{4}(\widehat{d}_t+\widehat{t}^*_{h,t}-\widehat{t}_{h,t})+\delta(\frac{2-v}{2}\phi^A_{1,t}-\frac{v}{2}\phi^A_{2,t})+\frac{v}{2}\phi^A_{5,t}=0 && (8.337)\\
\widehat{t}_{h,t} &: \quad -\frac{v(2-v)}{4}(\widehat{d}_t+\widehat{t}^*_{h,t}-\widehat{t}_{h,t})+\delta\frac{v}{2}\phi^A_{1,t}+\delta\frac{v}{2}\phi^A_{2,t}-\frac{v}{2}\phi^A_{5,t}=0 && (8.338)\\
\widehat{t}^*_{h,t} &: \quad \frac{v(2-v)}{4}(\widehat{d}_t+\widehat{t}^*_{h,t}-\widehat{t}_{h,t})+\delta\frac{2-v}{2}\phi^A_{1,t}+\delta\frac{2-v}{2}\phi^A_{2,t}-\frac{2-v}{2}\phi^A_{5,t}=0 && (8.339)\\
\widehat{t}_{f,t} &: \quad \frac{v(2-v)}{4}(\widehat{d}^*_t+\widehat{t}_{f,t}-\widehat{t}^*_{f,t})+\delta\frac{2-v}{2}\phi^A_{4,t}+\delta\frac{2-v}{2}\phi^A_{3,t}+\frac{2-v}{2}\phi^A_{5,t}=0 && (8.340)\\
\widehat{t}^*_{f,t} &: \quad -\frac{v(2-v)}{4}(\widehat{d}^*_t+\widehat{t}_{f,t}-\widehat{t}^*_{f,t})+\delta\frac{v}{2}\phi^A_{4,t}+\delta\frac{v}{2}\phi^A_{3,t}+\frac{v}{2}\phi^A_{5,t}=0 && (8.341)\\
\widehat{\tau}_t &: \quad \delta(\phi^A_{1,t}+\phi^A_{2,t})=0 && (8.342)\\
\widehat{\tau}^*_t &: \quad \delta(\phi^A_{3,t}+\phi^A_{4,t})=0 && (8.343)
\end{aligned}$$

Put (8.342) and (8.343) into the first two FOCs, we have

$$\begin{aligned}
\widehat{Y}_t-\theta_t-\phi^A_{5,t} &= 0 && (8.344)\\
\widehat{Y}^*_t-\theta^*_t+\phi^A_{5,t} &= 0 && (8.345)
\end{aligned}$$

Together with above equations (8.338)-(8.341), we have

$$\begin{aligned}
(\widehat{d}_t+\widehat{t}^*_{h,t}-\widehat{t}_{h,t}) &= 0 && (8.346)\\
(\widehat{d}^*_t+\widehat{t}_{f,t}-\widehat{t}^*_{f,t}) &= 0 && (8.347)
\end{aligned}$$

Above equation (8.336)-(8.337) imply $\phi^A_{3,t} = \phi^A_{1,t} = 0$ and final solution is

$$\phi^A_{1,t} = \phi^A_{2,t} = \phi^A_{3,t} = \phi^A_{4,t} = \phi^A_{5,t} = 0$$
$$\pi_{hh,t} = \pi^*_{hf,t} = \pi^*_{ff,t} = \pi_{fh,t} = 0$$
$$\widehat{Y}_t - \theta_t = 0; \widehat{Y}^*_t - \theta^*_t = 0$$
$$\widehat{t}_{h,t} = \widehat{t}^*_{h,t} = -\widehat{\tau}_t$$
$$\widehat{t}_{f,t} = \widehat{t}^*_{f,t} = -\widehat{\tau}^*_t$$
$$\widehat{d}^*_t = 0; \widehat{d}_t = \widehat{S}_t = 0$$

8.3 non-cooperation game $\rho = 1$

In non-cooperation game, Home loss function is:

$$(P7) \quad : \quad L_{h,0} = -W_{h,0} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ -(\widehat{C}_t - \widehat{L}_t) + \frac{1}{2}\widehat{L}^2_t \right\}$$
$$= E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t - \theta_t)^2 \\ \frac{v(2-v)}{8}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t})^2 \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi^2_{hh,t} + \frac{(2-v)}{2}\pi^2_{fh,t}\right] \\ -\frac{(2-v)}{4}\widehat{t}^2_{f,t} + \frac{(2-v)}{4}(\widehat{t}^*_{h,t})^2 \\ +\frac{(2-v)}{2}\left(-\widehat{\tau}_t + \widehat{\tau}^*_t - \widehat{t}^*_{h,t} + \widehat{t}_{f,t}\right) \\ +\frac{(2-v)}{4}\left[(\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{f,t} - \widehat{t}^*_{f,t}) + \widehat{\tau}^*_t\right]^2 \\ -\frac{(2-v)}{4}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right]^2 \end{array} \right\} \quad (8.348)$$

Minimize loss function by choosing $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{h,t}, \widehat{t}_{f,t}, \widehat{\tau}_t\right\}$, given $\left\{\widehat{Y}^*_t - \theta^*_t, \pi^*_{hf,t}, \pi^*_{ff,t}, \widehat{t}^*_{h,t}, \widehat{t}_{f,t}, \widehat{d}^*_t, \widehat{\tau}^*_t\right\}$, subject to

$$\pi_{hh,t} = \delta\left[\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{t}_{h,t} + \widehat{\tau}_t\right] + \beta E_t \pi_{hh,t+1} \quad (8.349)$$
$$\pi_{fh,t} = \delta\left[\widehat{Y}^*_t - \theta^*_t - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{f,t} - \widehat{t}^*_{f,t}) + \widehat{t}_{f,t} + \widehat{\tau}^*_t\right] + \beta E_t \pi_{fh,t+1} \quad (8.350)$$

$$\widehat{Y}_t - \widehat{Y}^*_t = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}^*_t + \widehat{q}_t - \frac{v}{2}(\widehat{t}_{h,t} - \widehat{t}^*_{f,t}) + \frac{2-v}{2}(\widehat{t}_{f,t} - \widehat{t}^*_{h,t}) \quad (8.351)$$

- Let $\varpi^A_{1,t}, \varpi^A_{2,t}, \varpi^A_{3,t}$ to be lagrangian multipliers for equations (8.349)-(8.351). First order conditions are:

$$\widehat{Y}_t - \theta_t : \widehat{Y}_t - \theta_t - \frac{(2-v)}{2}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right] + \delta\varpi^A_{1,t} - \varpi^A_{3,t} = 0 \tag{8.352}$$

$$\pi_{hh,t} : \frac{v\lambda}{2\delta}\pi_{hh,t} - \varpi^A_{1,t} + \varpi^A_{1,t-1} = 0 \tag{8.353}$$

$$\pi_{fh,t} : \frac{(2-v)\lambda}{2\delta}\pi_{fh,t} - \varpi^A_{2,t} + \varpi^A_{2,t-1} = 0 \tag{8.354}$$

$$\widehat{d}_t : \frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \frac{(2-v)v}{4}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right] \tag{8.355}$$

$$+\delta\frac{2-v}{2}\varpi^A_{1,t} + \frac{v}{2}\varpi^A_{3,t} = 0$$

$$\widehat{t}_{h,t} : -\frac{v(2-v)}{4}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) - \frac{(2-v)v}{4}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right] \tag{8.356}$$

$$+\delta\frac{v}{2}\varpi^A_{1,t} - \frac{v}{2}\varpi^A_{3,t} = 0$$

$$\widehat{t}_{f,t} : -\frac{(2-v)}{2}\widehat{t}_{f,t} - \frac{(2-v)}{2} - \frac{v(2-v)}{4}\left[(\widehat{Y}^*_t - \theta^*_t) - \frac{v}{2}(\widehat{d}^*_t + \widehat{t}_{f,t} - \widehat{t}^*_{f,t}) + \widehat{\tau}^*_t\right] \tag{8.357}$$

$$+\delta\frac{2-v}{2}\varpi^A_{2,t} + \frac{2-v}{2}\varpi^A_{3,t} = 0$$

$$\widehat{\tau}_t : -\frac{(2-v)}{2} - \frac{(2-v)}{2}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right] + \delta\varpi^A_{1,t} = 0 \tag{8.358}$$

The Foreign loss function is:

$$(P7): L_{h,0}^{*} = E_t \sum_{t=0}^{\infty} (\beta)^t \left\{ \begin{array}{c} \frac{1}{2}(\widehat{Y}_t^{*} - \theta_t^{*})^2 \\ +\frac{v(2-v)}{8}(\widehat{d}_t^{*} + \widehat{t}_{f,t} - \widehat{t}_{f,t}^{*})^2 \\ +\frac{\lambda}{2\delta}\left[\frac{v}{2}\pi_{ff,t}^{*2} + \frac{(2-v)}{2}\pi_{hf,t}^{*2}\right] \\ +\frac{(2-v)}{4}\left[\left(\widehat{t}_{f,t}\right)^2 - \left(\widehat{t}_{h,t}^{*}\right)^2\right] \\ -\frac{(2-v)}{2}\left(-\widehat{\tau}_t + \widehat{\tau}_t^{*} - \widehat{t}_{h,t}^{*} + \widehat{t}_{f,t}\right) \\ -\frac{(2-v)}{4}\left[(\widehat{Y}_t^{*} - \theta_t^{*}) - \frac{v}{2}(\widehat{d}_t^{*} + \widehat{t}_{f,t} - \widehat{t}_{f,t}^{*}) + \widehat{\tau}_t^{*}\right]^2 \\ +\frac{(2-v)}{4}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^{*} - \widehat{t}_{h,t}) + \widehat{\tau}_t\right]^2 \end{array} \right\} \tag{8.359}$$

Minimize loss function by choosing $\left\{\widehat{Y}_t^{*} - \theta_t^{*}, \pi_{hf,t}^{*}, \pi_{ff,t}^{*}, \widehat{t}_{h,t}^{*}, \widehat{t}_{f,t}, \widehat{d}_t^{*}, \widehat{\tau}_t^{*}\right\}$, given $\left\{\widehat{Y}_t - \theta_t, \pi_{hh,t}, \pi_{fh,t}, \widehat{d}_t, \widehat{t}_{h,t}, \widehat{t}_{f,t}, \widehat{\tau}_t\right\}$, subject to

$$\pi_{ff,t}^{*} = \delta\left[\widehat{Y}_t^{*} - \theta_t^{*} + \frac{2-v}{2}(\widehat{d}_t^{*} + \widehat{t}_{f,t} - \widehat{t}_{f,t}^{*}) + \widehat{t}_{f,t}^{*} + \widehat{\tau}_t^{*}\right] + \beta E_t \pi_{ff,t+1}^{*} \tag{8.360}$$

$$\pi_{hf,t}^{*} = \delta\left[\widehat{Y}_t - \theta_t - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^{*} - \widehat{t}_{h,t}) + \widehat{t}_{h,t}^{*} + \widehat{\tau}_t\right] + \beta E_t \pi_{hf,t+1}^{*} \tag{8.361}$$

$$\widehat{Y}_t - \widehat{Y}_t^{*} = \frac{v}{2}\widehat{d}_t - \frac{v}{2}\widehat{d}_t^{*} + \widehat{q}_t - \frac{v}{2}(\widehat{t}_{h,t} - \widehat{t}_{f,t}^{*}) + \frac{2-v}{2}(\widehat{t}_{f,t} - \widehat{t}_{h,t}^{*}) \tag{8.362}$$

- Let $\varpi_{1,t}^{A*}, \varpi_{2,t}^{A*}, \varpi_{3,t}^{A*}$ to be lagrangian multipliers for equations (8.360)-(8.362). First order conditions are:

$$\widehat{Y}_t^* - \theta_t^* : \widehat{Y}_t^* - \theta_t^* - \frac{(2-v)}{2}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{\tau}_t^*\right] + \delta\varpi_{1,t}^{A*} + \varpi_{3,t}^{A*} = 0 \tag{8.363}$$

$$\pi_{ff,t}^* : \frac{v\lambda}{2\delta}\pi_{ff,t}^* - \varpi_{1,t}^{A*} + \varpi_{1,t-1}^{A*} = 0 \tag{8.364}$$

$$\pi_{hf,t}^* : \frac{(2-v)\lambda}{2\delta}\pi_{hf,t}^* - \varpi_{2,t}^{A*} + \varpi_{2,t-1}^{A*} = 0 \tag{8.365}$$

$$\widehat{d}_t^* : \frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \frac{(2-v)v}{4}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{\tau}_t^*\right] \tag{8.366}$$

$$+\delta\frac{2-v}{2}\varpi_{1,t}^{A*} - \frac{v}{2}\varpi_{3,t}^{A*} = 0$$

$$\widehat{t}_{f,t}^* : -\frac{v(2-v)}{4}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) - \frac{(2-v)v}{4}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{\tau}_t^*\right] \tag{8.367}$$

$$+\delta\frac{v}{2}\varpi_{1,t}^{A*} + \frac{v}{2}\varpi_{3,t}^{A*} = 0$$

$$\widehat{t}_{h,t}^* : -\frac{(2-v)}{2}\widehat{t}_{h,t}^* - \frac{(2-v)}{2} - \frac{v(2-v)}{4}\left[(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t}) + \widehat{\tau}_t\right] \tag{8.368}$$

$$+\delta\frac{2-v}{2}\varpi_{2,t}^{A*} - \frac{2-v}{2}\varpi_{3,t}^{A*} = 0$$

$$\widehat{\tau}_t^* : -\frac{(2-v)}{2} - \frac{(2-v)}{2}\left[(\widehat{Y}_t^* - \theta_t^*) - \frac{v}{2}(\widehat{d}_t^* + \widehat{t}_{f,t} - \widehat{t}_{f,t}^*) + \widehat{\tau}_t^*\right] + \delta\varpi_{1,t}^{A*} = 0 \tag{8.369}$$

- Solution:

(8.355) and (8.356) imply $\varpi_{1,t}^A = 0$.
(8.358) could be simplified as

$$(\widehat{Y}_t - \theta_t) - \frac{v}{2}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t}) + \widehat{\tau}_t = -1 \tag{8.370}$$

(8.352) and (8.355) could be simplified as

$$\widehat{Y}_t - \theta_t = -\frac{(2-v)}{2}(\widehat{d}_t + \widehat{t}_{h,t}^* - \widehat{t}_{h,t}) = -\frac{(2-v)}{2} + \varpi_{3,t}^A \tag{8.371}$$

And $\varpi_{1,t}^A = 0$ implies $\pi_{hh,t} = 0$. Which means

$$\widehat{Y}_t - \theta_t + \frac{2-v}{2}(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) + \widehat{t}_{h,t} + \widehat{\tau}_t = 0 \tag{8.372}$$

The above three equations help to derive

$$\begin{aligned}
(\widehat{d}_t + \widehat{t}^*_{h,t} - \widehat{t}_{h,t}) - \widehat{\tau}_t - 1 &= 0 && (8.373)\\
\widehat{t}_{h,t} + \widehat{\tau}_t &= 0 && (8.374)\\
\widehat{d}_t + \widehat{t}^*_{h,t} &= 1 && (8.375)\\
\widehat{Y}_t - \theta_t &= -\frac{(2-v)}{2}(\widehat{\tau}_t + 1) && (8.376)\\
\boldsymbol{\varpi}^A_{3,t} &= -\frac{(2-v)}{2}\widehat{\tau}_t && (8.377)
\end{aligned}$$

Immediately, we find that if the flexible price equilibrium is replciated, $\widehat{\tau}_t = -1$. The solutions of Foreign country is similar, together with equation (8.357), we have

$$\begin{aligned}
\widehat{t}_{f,t} &= 1 + \frac{v}{2} + \delta\frac{2-v}{2}\varpi^A_{2,t} - \frac{(2-v)}{2}\widehat{\tau}_t && (8.378)\\
&= 2 + \delta\frac{2-v}{2}\varpi^A_{2,t} && (8.379)
\end{aligned}$$

Together with (8.350), we have

$$\begin{aligned}
\pi_{fh,t} &= \delta\left[\delta\frac{2-v}{2}\varpi^A_{2,t} + 1\right] + \beta E_t \pi_{fh,t+1}\\
&\Longrightarrow\\
\frac{2\delta}{(2-v)\lambda}(\varpi_{2,t} - \varpi_{2,t-1}) &= \delta\left[\delta\frac{2-v}{2}\varpi^A_{2,t} + 1\right] + \beta E_t\left(\frac{2\delta}{(2-v)\lambda}(\varpi_{2,t+1} - \varpi_{2,t})\right)
\end{aligned}$$

Due to the existence of constant term, we find that $\varpi^A_{2,t} \neq 0$. In other word, flexible price equilibrium cannot be achived under Nash game setting.