Appendix

Appendiix A analyzes long difference estimators. Appendix B reports descriptive statistics for the empirical setting. Appendix C estimates the effects of climate change by industry. Appendix D derives the indirect least squares estimator when agents observe forecasts but the empirical researcher does not. Appendix E contains proofs.

A Long Difference Estimators

Recognizing the difficulty of accounting for adaptation, some empirical literature averages outcomes over long timesteps, a procedure known as "long differences" (e.g., Dell et al., 2012; Burke and Emerick, 2016).[48] In order to obtain sharper results, assume temporarily that specialized forecasts are available only one period in advance and that Σ is diagonal. I use Σ_{ij} to indicate element (i,j) of Σ. Define

$$\breve{\pi}_s \triangleq \frac{1}{\Delta} \sum_{t=s}^{s+\Delta-1} \pi_t$$

as average payoffs over Δ timesteps beginning with $t=s$. Define $\breve{w}_s$ and $\breve{f}_{1,s}$ analogously. Consider the following regression:

$$\breve{\pi}_{js} = \breve{\alpha}_j + \breve{\Lambda}\breve{w}_{js} + \breve{\lambda}\breve{f}_{j1,s} + \breve{\eta}_{js},$$

with observations only every Δ timesteps (i.e., no overlap in averaging intervals). The next proposition shows that estimating this regression does not generally get us closer to the effect of climate than did estimating regression (15) with Δ lags:

Proposition A-1 (Long Differences)**.** *Let Assumption 1 hold, or let Assumptions 2 and 3 hold. Also, let Σ be diagonal and $\Sigma_{33}=0$. Then:*

$$\text{plim}\left(\hat{\breve{\Lambda}} + \hat{\breve{\lambda}}\right) = \text{plim} \sum_{i=0}^{\Delta-1} \frac{\Delta - i}{\Delta}\left[\hat{\Lambda}_i + \hat{\lambda}_i\right] + \text{plim}\, \frac{1}{\Delta}\sum_{i=0}^{\Delta-1} \frac{\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}\hat{\lambda}_{i+1}. \tag{A-1}$$

If $\Psi > 0$, then

$$\text{plim}\left(\hat{\breve{\Lambda}} + \hat{\breve{\lambda}}\right) = \bar{\pi}_w + \breve{\omega}\left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right),$$

where $\breve{\omega} \in (0, \omega_\Delta)$, with $\omega_\Delta \in (0,\omega)$ from Corollary 3 and ω and Ω from Proposition 2.

Proof. See Appendix E.14. □

[48]The subsequent analysis does not depend on whether the operation is summing or averaging.

Even though equation (A-1) does not explicitly include lags on the left-hand side, the estimated coefficients $\hat{\mathring{\Lambda}}$ and $\hat{\mathring{\lambda}}$ do incorporate effects of lagged weather and lagged forecasts owing to correlations between payoffs and lagged weather and forecasts within a timestep (see also Ghanem and Smith, 2021). As a result, $\hat{\mathring{\Lambda}} + \hat{\mathring{\lambda}}$ bears some resemblance to summing Δ lags from regression (15). However, only observations at the very end of each long timestep have a full Δ lags within the same timestep. All other observations have fewer than Δ lags implicitly estimated. Summing the long difference coefficients is therefore analogous to summing downweighted versions of the lag coefficients from regression (15).

As in Corollary 3, the bias introduced by $\breve{\omega}$ is particularly easy to sign when $\Psi > 0$. In this case, Corollary 3 showed that summing the first Δ lags amplified the bias from historical restraints relative to summing infinite lags. We now see that implicitly summing these lags through long timesteps further amplifies that bias because nearly all observations within the long timestep have fewer than Δ lags. Long differences are not generally superior to simply estimating a standard panel model with Δ lags and summing the coefficients.[49]

Researchers sometimes compare long difference estimates to standard panel estimates in order to learn whether whether short-run adaptation differs from long-run adaptation. The hope is that the long difference estimator is identified by spatially heterogeneous rates of climate change that manifest over decades. However, long difference estimators are in fact identified in the foregoing analysis even though there is, by construction, no climate change in the present setting (C is here constant over agents and over time). In fact, they are identified by random differences in sequences of the same transient weather shocks that identify panel estimators such as (15). This source of identification is unavoidable in applications, whether or not there is also variation in C. At best, long difference estimators conflate the identifying variation of transient weather shocks with differential rates of climate change, but at worst, they capture nothing but this familiar identifying variation. We should judge the latter case to be especially likely when long difference and panel estimators produce similar results, as in fact has been reported in previous work (summarized in Hsiang, 2016).

B Descriptive Statistics

Table A-1 reports descriptive statistics for the empirical application, broken down by Census Region.

[49]Comparing long difference estimates to panel estimates with few lags does tell us something about the importance of historical restraints ($\breve{\omega}$ vs $\omega_{I'}$), which relates to the difference between long-run and short-run adaptation, but so too would simply changing the number of lags used, per Corollary 3.

Table A-1: Descriptive Statistics

	Midwest		Northeast		South		West	
	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev
Output ($Billion)	3.1	13.6	15.2	40.4	3.9	14.2	9.1	37.7
Population (Thous)	63.4	211.9	254.8	375.8	81.8	215.7	169.2	623.2
Output p.c. ($Thous)	40.6	20.1	45.3	25.5	59.7	707.1	48.6	56.2
Temperature (deg C)	9.8	2.9	9.2	2.2	16.6	2.9	9.3	4.1

Years: 2001–2019, excluding 2007 and 2008.
Output is annual, in year 2012 dollars.
Temperature is the unweighted average across counties.

C Industry-Level Results

This appendix extends the primary empirical specification from the main text to assess effects by industry. Data are again from the Bureau of Economic Analysis, except now for each industry's output by county. I group industries using definitions from Table A2 in Colacito et al. (2019). Climate change calculations use the projected national change in temperature, of 4.0°C. Each industry is run in a separate regression. The only changes to the estimating equations are that they now estimate a single coefficient across Census regions and that the log-output weights are industry-specific.[50]

Figure A-1 reports results from the ILS and OLS estimators. Both estimators project losses across a number of industries, including "Agriculture, forestry, fishing", "Mining", and "Utilities". Many of these are intuitively exposed to weather. The ILS estimator tends to be noisier but does produce solidly more negative point estimates in several important cases ("Mining", "Transportation", "Utilities"). Direct effects again drive the ILS estimates.

Figure A-2 reports the estimated Ψ for each industry. All of the point estimates are positive, although many are noisy. Finding positive Ψ is consistent with the main text's results.

D Analysis When Agents Observe Forecasts but the Empirical Researcher Does Not

Consider the following regression, which uses leads of weather as proxies for forecasts:

$$\pi_{jt} = \alpha_j + \sum_{i=-2}^{2} \Phi_i w_{j(t-i)} + \eta_{jt}. \tag{A-2}$$

[50]Results exclude "Finance, insurance, and real estate" due to convergence problems in Stata's `nlcom` and exclude "Communication/Information", "Retail", and "Wholesale" because estimates are rather noisy.

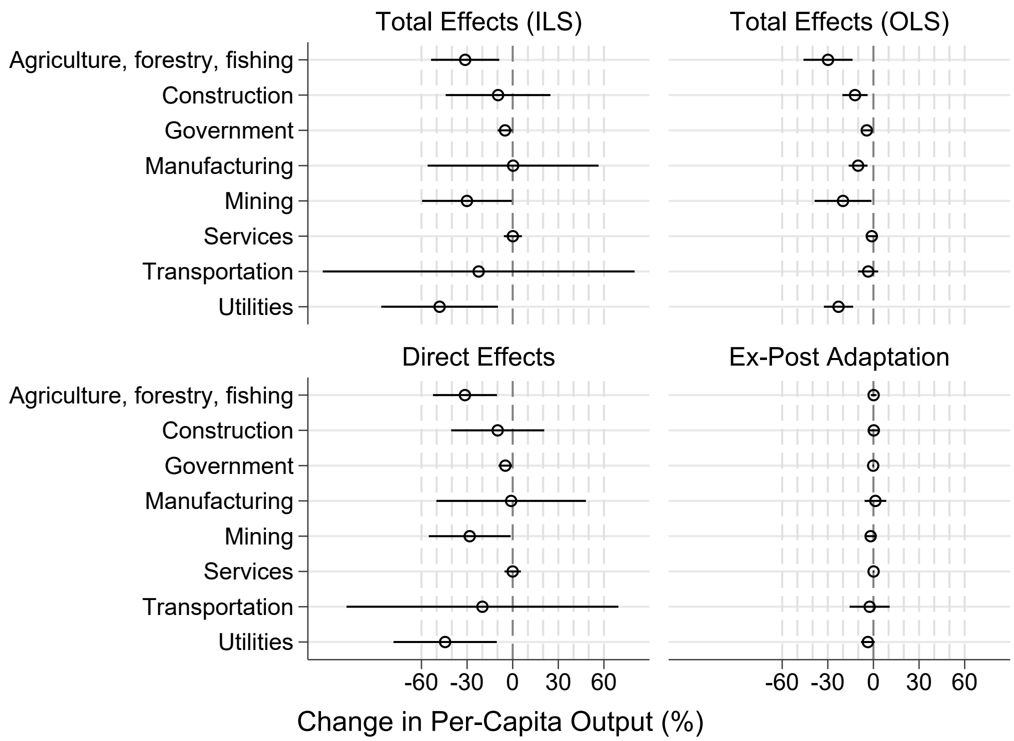

Figure A-1: Projected end-of-century percentage change in industry output per capita due to business-as-usual climate change's effect on temperature.

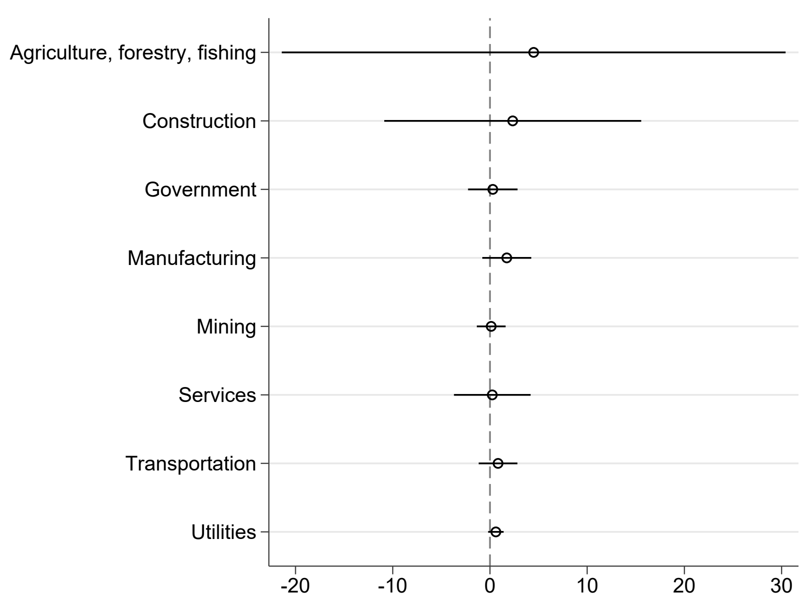

Figure A-2: Estimated Ψ by industry.

The right-hand side contains only the fixed effect, the contemporary effect of weather, two lags of weather, and two leads of weather. I use Σ_{ij} to indicate element (i, j) of Σ. The following proposition presents the indirect least squares estimator for climate impacts:

Proposition A-2 (Indirect Least Squares With Unobserved Forecasts)**.** *Let Assumption 1 hold, or let Assumptions 2 and 3 hold. Consider estimating regression* (A-2)*, assuming that* $\text{plim}\,\hat{\Phi}_1, \hat{\Phi}_{-1} \neq 0$ *and* $\text{plim}\,\hat{\Phi}_2/\hat{\Phi}_1 \neq 1/\beta$*. If* Σ *is diagonal, then:*

$$\begin{aligned}
&\text{plim}\Bigg(\overbrace{\hat{\Phi}_0 - \frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} + \frac{1}{\beta}\hat{\Phi}_{-1} + \frac{1}{\beta^2}\hat{\Phi}_{-2}}^{\text{direct effects}} \\
&\overbrace{- \frac{1-\beta}{\beta}\left[\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} \right]}^{\text{ex-post adaptation}} \\
&\overbrace{- \frac{1-\beta}{\beta}\left[\hat{\Phi}_{-1} - \left(\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta} \right)\hat{\Phi}_{-2} \right]}^{\text{ex-ante adaptation (estimated)}} \frac{1}{\Sigma_{22}/trace(\Sigma)} \\
&\overbrace{+ \frac{1-\beta}{\beta}\frac{\hat{\Phi}_{-2}}{\hat{\Phi}_{-1}}\left[\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} \right]}^{\text{ex-ante adaptation (Ω adjustment)}} \frac{\Sigma_{22}/trace(\Sigma)}{\Sigma_{33}/trace(\Sigma)} \Bigg) \\
=&\bar{\pi}_w + \tilde{\omega}\left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g} \right] \frac{\partial \bar{A}(K, C)}{\partial C},
\end{aligned} \tag{A-3}$$

and

$$\Psi \propto \text{plim}\,\frac{\hat{\Phi}_2}{\hat{\Phi}_1}.$$

If $\Psi > 0$*, then* $\tilde{\omega} < 1$*. If* $\Psi = 0$*, then* $\tilde{\omega} = 1$*. If* $\Psi < 0$*, then* $\tilde{\omega} > 1$*.*

Proof. See Appendix E.15. □

The intuition for identification is as presented following Proposition 5, but now we have to adjust for forecasts acting as omitted variables. This has a few consequences. Most are relatively minor, but one can be important in some applications. First, the derivation requires that weather be serially uncorrelated. This assumption limits the degree of omitted variables bias in each reduced-form coefficient. Second, all covariates (other than w_{t+2}) are correlated with some forecast issued prior to time t and that affects π_{jt}. This correlation introduces additional ex-ante adaptation terms into each coefficient, and we need to subtract these off to recover the effects of interest. The expressions are therefore messier than in Proposition 5.[51] Third, we no longer have forecasts to identify the adjustment to the ex-post adaptation terms seen in Proposition 5, so we now need to calibrate β in order to make the required adjustment. This requirement is not too onerous since we needed a value for β to calculate climate impacts in (17) anyway.

Most importantly, our estimate of ex-ante adaptation from the coefficient $\hat{\Phi}_{-1}$ on the lead of weather tends to be too small in magnitude: $\hat{\Phi}_{-1}$ reflects the total variation in weather, but only a fraction $\Sigma_{22}/trace(\Sigma)$ of that variation was forecasted one period ahead of time. The bias from proxying forecasts with the lead of weather vanishes as the fraction goes to 1. In contrast, if time $t+1$ weather is largely unknown at time t, then we estimate very little ex-ante adaptation even though an agent might undertake substantial ex-ante adaptation to climate change. (Analogous comments apply to the estimated correction factor for Ω.)

With a calibrated Σ in hand, we are left with the same biases as in Proposition 5, involving having fixed K and the adjustments due to $\tilde{\omega}$ (which we again sign using the estimated sign of Ψ).

E Formal Analysis and Proofs

E.1 Deriving equation (6)

With A_t defined implicitly from the first-order condition $\pi_A = 0$, approximate A_t around $w_t = C$ and use either Assumption 1 or Assumption 2:

$$A_t = \bar{A} + \frac{\bar{\pi}_{wA}}{-\bar{\pi}_{AA}}(w_t - C). \tag{A-4}$$

Therefore,

$$E_0[A_t] = \bar{A}.$$

Approximating the payoff function around $w_t = C$ and using either Assumption 1 or Assumption 2, we have:

$$\begin{aligned} E_0[\pi(w_t, A_t, S_t; K)] =& \bar{\pi} + \bar{\pi}_w \underbrace{(E_0[w_t] - C)}_{=0} + \bar{\pi}_A \underbrace{(E_0[A_t] - \bar{A})}_{=0} \\ &+ \frac{1}{2}\bar{\pi}_{ww} E_0[(w_t - C)^2] + \frac{1}{2}\bar{\pi}_{AA} E_0[(A_t - \bar{A})^2] + \bar{\pi}_{wA} Cov_0[A_t, w_t], \end{aligned} \tag{A-5}$$

for $t > 2$. Differentiating with respect to C, applying either Assumption 1 or Assumption 2 again, and using that these assumptions imply $\bar{A} = E_0[A_t]$, we have the expression given in the text.

[51] If forecasts are available to the agent only one period ahead, then the terms with $\hat{\Phi}_{-2}$ vanish. If forecasts are available more than two periods ahead, then results generalize straightforwardly.

E.2 Proof of Proposition 1

Following the derivation of equation (A-5) and applying the first-order condition, we have:

$$\pi(w_t, A_t, S_t; K) = \bar{\pi} + \bar{\pi}_w(w_t - C) + \frac{1}{2}\bar{\pi}_{ww}(w_t - C)^2 + \frac{1}{2}\bar{\pi}_{AA}(A_t - \bar{A})^2 + \bar{\pi}_{wA}(w_t - C)(A_t - \bar{A}).$$

Therefore,

$$\begin{aligned} Cov[\pi_{jt}, w_{jt} - C] =& \bar{\pi}_w \zeta^2 trace(\Sigma) + \frac{1}{2}\bar{\pi}_{ww} Cov[w_{jt} - C, (w_{jt} - C)^2] \\ &+ \frac{1}{2}\bar{\pi}_{AA} Cov[w_{jt} - C, (A_{jt} - \bar{A})^2] + \bar{\pi}_{wA} Cov[w_{jt} - C, (w_{jt} - C)(A_{jt} - \bar{A})] \\ =& \bar{\pi}_w \zeta^2 trace(\Sigma) + \frac{1}{2}\bar{\pi}_{ww} Cov[w_{jt} - C, (w_{jt} - C)^2] + \frac{1}{2}\bar{\pi}_{AA} Cov[w_{jt} - C, (A_{jt} - \bar{A})^2] \\ &+ \bar{\pi}_{wA} Cov[w_{jt} - C, w_{jt} A_{jt}] - \bar{A}\bar{\pi}_{wA} Var[w_{jt}] - C\bar{\pi}_{wA} Cov[w_{jt} - C, A_{jt}]. \end{aligned}$$

If Assumption 3 holds, then $Cov[w_{jt} - C, (w_{jt} - C)^2] = 0$, or if Assumption 1 holds, then $Cov[w_{jt} - C, (w_{jt} - C)^2] \approx 0$ because it is of order ζ^3. Using results from Bohrnstedt and Goldberger (1969),

$$Cov[w_{jt}, w_{jt}A_{jt}] = E[A_{jt}]\, Var[w_{jt}] + C\, Cov[A_{jt}, w_{jt}] + E[(w_{jt} - C)^2(A_{jt} - E[A_t])].$$

If either Assumption 3 or Assumption 1 holds, then this becomes:

$$Cov[w_{jt}, w_{jt}A_{jt}] = E[A_{jt}]\, Var[w_{jt}] + C\, Cov[A_{jt}, w_{jt}].$$

Using $E[A_{jt}] = \bar{A}$, we find:

$$Cov[\pi_{jt}, w_{jt} - C] = \bar{\pi}_w \zeta^2 trace(\Sigma) + \frac{1}{2}\bar{\pi}_{AA} Cov[w_{jt} - C, (A_{jt} - \bar{A})^2].$$

Using (A-4), observe that:

$$Cov[w_{jt} - C, (A_{jt} - \bar{A})^2] = \left(\frac{\bar{\pi}_{wA}}{-\bar{\pi}_{AA}}\right)^2 Cov[w_{jt} - C, (w_{jt} - C)^2],$$

which is 0 if either Assumption 1 or Assumption 3 holds. We therefore have:

$$Cov[\pi_{jt}, w_{jt} - C] = \bar{\pi}_w \zeta^2 trace(\Sigma).$$

The result follows from observing that $Var[w_{jt} - C] = \zeta^2 trace(\Sigma)$.

E.3 Proof that there is a unique maximizer in the deterministic model ($\zeta = 0$)

With $\zeta = 0$, rewrite payoffs as a function of S_t and S_{t+1} by using $A_t = h^{-1}(S_{t+1} - gS_t)$: $\tilde{\pi}(S_t, S_{t+1}) \triangleq \pi(C, A_t, S_t; K)$. If the payoff function is strictly concave and bounded, then there is a unique maximizer by Theorem 4.8 in Stokey and Lucas (1989). Strict concavity requires that $\tilde{\pi}_{S_t S_t} < 0$ and $\tilde{\pi}_{S_t S_t}\tilde{\pi}_{S_{t+1} S_{t+1}} - (\tilde{\pi}_{S_t S_{t+1}})^2 > 0$. We have:

$$\begin{aligned}\tilde{\pi}_{S_t S_t}\tilde{\pi}_{S_{t+1} S_{t+1}} - (\tilde{\pi}_{S_t S_{t+1}})^2 =&(1/h')^4 \left[(h')^2\pi_{SS} + 2h'\pi_{AS} + \pi_{AA} - (h''/h')\pi_A\right]\left[\pi_{AA} - (h''/h')\pi_A\right] \\ &- (1/h')^4 \left[h'\pi_{AS} + \pi_{AA} - (h''/h')\pi_A\right]^2 \\ =&(1/h')^4 \left[(h')^2\pi_{SS}\right]\left[\pi_{AA} - (h''/h')\pi_A\right] - (1/h')^4 (h'\pi_{AS})^2.\end{aligned}$$

This is strictly positive if and only if inequality (1) holds. By the inequality of arithmetic and geometric means, inequality (1) in turn implies

$$h'\pi_{AS} < \frac{1}{2}\left(-\pi_{AA} + (h''/h')\pi_A\right) - \frac{1}{2}(h')^2\pi_{SS},$$

which is equivalent to $\tilde{\pi}_{S_t S_t} < 0$. We have therefore established that inequality (1) implies that payoffs are strictly concave in S_t and S_{t+1}.

E.4 Proof that deterministic model ($\zeta = 0$) has a unique steady state and is saddle-path stable

Fix $\zeta = 0$, in which case $w_t = f_{1,t} = f_{2,t} = C$ at all times t.

The first-order condition for the deterministic model is:

$$0 = \pi_A(C, A_t, S_t; K) + \beta h'(A_t)\, V_S(S_{t+1}, C, C, C; 0, K).$$

This implies:

$$V_S(S_{t+1}, C, C, C; 0, K) = \frac{-\pi_A(C, A_t, S_t; K)}{\beta h'(A_t)}.$$

The envelope theorem yields:

$$V_S(S_{t+1}, C, C, C; 0, K) = \pi_S(C, A_{t+1}, S_{t+1}; K) + \beta\, g\, V_S(S_{t+2}, C, C, C; 0, K).$$

Advancing the first-order condition by one timestep and substituting in, we have the Euler equation:

$$-\pi_A(C, A_t, S_t; K) = \beta\, h'(A_t)\, \pi_S(C, A_{t+1}, S_{t+1}; K) + \beta\, h'(A_t)\, g\, \frac{-\pi_A(C, A_{t+1}, S_{t+1}; K)}{h'(A_{t+1})}. \tag{A-6}$$

The steady state (denoted with a bar) is defined by the following pair of equations:

$$-\pi_A(C,\bar{A},\bar{S};K) = \beta\, h'(\bar{A})\,\pi_S(C,\bar{A},\bar{S};K) - \beta\, g\,\pi_A(C,\bar{A},\bar{S};K),$$
$$\bar{S} = g\bar{S} + h(\bar{A}).$$

The second implies:

$$\bar{S} = \frac{h(\bar{A})}{1-g}. \tag{A-7}$$

Substituting into the first equation and rearranging, we have:

$$-(1-\beta g)\pi_A\left(C,\bar{A},\frac{h(\bar{A})}{1-g};K\right) - \beta h'(\bar{A})\pi_S\left(C,\bar{A},\frac{h(\bar{A})}{1-g};K\right) = 0. \tag{A-8}$$

The derivative of the left-hand side with respect to $\bar{A}$ is

$$-(1-\beta g)\bar{\pi}_{AA} - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \beta h''(\bar{A})\bar{\pi}_S - (1-\beta g)\frac{h'(\bar{A})}{1-g}\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}.$$

Substituting for $\beta\bar{\pi}_S$ from (A-6), this becomes:

$$(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \left[\frac{1-\beta g}{1-g} + \beta\right] h'(\bar{A})\bar{\pi}_{AS}.$$

This expression is strictly positive if and only if

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1-(1+\beta)g+\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta-2\beta g}. \tag{A-9}$$

From (5), we have

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1-2g(1+\beta)+3\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta-2\beta g}.$$

The right-hand side of this last inequality is weakly less than the right-hand side of inequality (A-9). Therefore inequality (A-9) holds, which in turn implies that a steady state exists by (3) and (4) and that this steady state is unique.

The Euler equation (A-6) implicitly defines $A^*_{t+1}(A_t,S_t)$. Using the implicit function theorem:

$$\frac{\partial A_{t+1}}{\partial S_t} = \frac{h'(A_{t+1})\left[-\frac{\pi_{AS}(C,A_t,S_t;K)}{h'(A_t)} - \beta g\pi_{SS}(C,A_{t+1},S_{t+1};K) + \beta g^2\,\frac{\pi_{AS}(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right]}{\beta h'(A_{t+1})\pi_{AS}(C,A_{t+1},S_{t+1};K) + \beta g\left(-\pi_{AA}(C,A_{t+1},S_{t+1};K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)},$$

$$\frac{\partial A_{t+1}}{\partial A_t} = \frac{h'(A_{t+1})\left[-\beta h'(A_t)\,\pi_{SS}(C,A_{t+1},S_{t+1};K) + \frac{-\pi_{AA}(C,A_t,S_t;K)}{h'(A_t)} + h''(A_t)\frac{\pi_A(C,A_t,S_t;K)}{[h'(A_t)]^2}\right]}{\beta h'(A_{t+1})\pi_{AS}(C,A_{t+1},S_{t+1};K) + \beta g\left(-\pi_{AA}(C,A_{t+1},S_{t+1};K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)}$$
$$+ \frac{h'(A_{t+1})\beta g h'(A_t)\,\frac{\pi_{AS}(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}}{\beta h'(A_{t+1})\pi_{AS}(C,A_{t+1},S_{t+1};K) + \beta g\left(-\pi_{AA}(C,A_{t+1},S_{t+1};K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)}.$$

Approximate A_{t+1} around the steady state:

$$\begin{aligned} A_{t+1} \approx & \bar{A} + \frac{-(1-\beta g^2)\bar{\pi}_{AS} - \beta g h'(\bar{A})\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}(S_t - \bar{S}) \\ & + \frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta \left[h'(\bar{A})\right]^2 \bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}(A_t - \bar{A}). \end{aligned}$$

Linearize the dynamic system around the steady state:

$$\begin{bmatrix} A_{t+1} - \bar{A} \\ S_{t+1} - \bar{S} \end{bmatrix} \approx \begin{bmatrix} \frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta \left[h'(\bar{A})\right]^2 \bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} & \frac{-(1-\beta g^2)\bar{\pi}_{AS} - \beta g h'(\bar{A})\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} \\ h'(\bar{A}) & g \end{bmatrix} \begin{bmatrix} A_t - \bar{A} \\ S_t - \bar{S} \end{bmatrix}.$$

The determinant is $1/\beta$, which is > 1. Therefore both eigenvalues have the same sign. The characteristic equation is

$$0 = z^2 - \left[\frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta \left[h'(\bar{A})\right]^2 \bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} + g \right] z + \frac{1}{\beta}.$$

This is a parabola that opens up. At $z = 1$, its value is:

$$-\frac{(1-g)(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - (1+\beta-2\beta g)h'(\bar{A})\bar{\pi}_{AS} - \beta \left[h'(\bar{A})\right]^2 \bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}.$$

By inequality (A-9), the numerator is positive. If the denominator is positive, then the expression is negative, so there is one root $\in (0, 1)$ and one root > 1, making the system saddle-path stable. If the denominator is negative, then the analogous expression for $z = -1$ is negative, so there is one root $\in (-1, 0)$ and one root < -1, making the system again saddle-path stable.

E.5 Optimal actions in the stochastic system

The first-order condition is:

$$0 = \pi_A(w_t, A_t, S_t; K) + \beta h'(A_t)\, E_t[V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K)].$$

This implies:

$$E_t[V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K)] = \frac{-\pi_A(w_t, A_t, S_t; K)}{\beta h'(A_t)}.$$

The envelope theorem yields:

$$V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K) = \pi_S(w_{t+1}, A_{t+1}, S_{t+1}; K) + \beta g E_{t+1}[V_S(S_{t+2}, w_{t+2}, f_{1,t+2}, f_{2,t+2}; \zeta, K)].$$

Advancing the first-order condition by one timestep and substituting in, we have the stochastic Euler equation:

$$\frac{-\pi_A(w_t, A_t, S_t; K)}{h'(A_t)} = \beta\, E_t[\pi_S(w_{t+1}, A_{t+1}, S_{t+1}; K)] + \beta\, g\, E_t\left[\frac{-\pi_A(w_{t+1}, A_{t+1}, S_{t+1}; K)}{h'(A_{t+1})}\right]. \tag{A-10}$$

For $\zeta = 0$, the weather in period $t+2$ matches the forecast $f_{2,t}$ and the weather is always C after period $t+2$. So we are back to the deterministic system in period $t+3$. Consider some distant time T at which the world ends. We will work backwards from there, solving for time $t+3$ policy as $T \to \infty$. Once we have that, we solve for time $t+2$ policy given $w_{t+2} = f_{2,t}$ and $f_{1,t+2} = f_{2,t+2} = C$; then we solve for time $t+1$ policy given $w_{t+1} = f_{1,t}$, $f_{1,t+1} = f_{2,t}$, and $f_{2,t+1} = C$; and finally we solve for time t policy given w_t, $f_{1,t}$, and $f_{2,t}$.

Write A_t as $A(S_t, w_t, f_{1,t}, f_{2,t}; \zeta)$ and define $\tilde{A}_t \triangleq A(S_t, w_t, f_{1,t}, f_{2,t}; 0)$. At time T, we have a static problem. The first-order condition is $\pi_A = 0$. Note that $\partial \tilde{A}_T / \partial S_T = \pi_{AS}/[-\pi_{AA}]$. Using the time $T-1$ Euler equation, first-order approximate $\tilde{A}_{T-1}$ around $S_{T-1} = \bar{S}$. This approximation is exact when either Assumption 1 or 2 holds and $(S_{T-1} - \bar{S})^2$ is small. We thereby obtain $\tilde{A}_{T-1}$ as a function of S_{T-1}:

$$\tilde{A}_{T-1} = \bar{A} + \frac{\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\frac{\bar{\pi}_{AS}}{-\bar{\pi}_{AA}}}{\chi_{T-1}}(S_{T-1} - \bar{S}),$$

where

$$\begin{aligned}\chi_{T-1} \triangleq& \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} \\ &- \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A})\left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\frac{\bar{\pi}_{AS}}{-\bar{\pi}_{AA}}.\end{aligned}$$

Denote the coefficient on $S_t - \bar{S}$ in $\tilde{A}_t$ as Z_t. Stepping backwards through time, we find the following relationships:

$$\begin{aligned}Z_t =& \frac{\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}}{\chi_t}, \\ \chi_t =& \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} \\ &- \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A})\left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}.\end{aligned}$$

Consider the fate of Z_t and χ_t as the terminal time T recedes to infinity. The steady state is:

$$\bar{Z} = \frac{\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\bar{Z}}{\bar{\chi}},$$

$$\bar{\chi} = \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A}) \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\bar{Z}.$$

Substitute $\bar{\chi}$ into $\bar{Z}$ and rearrange:

$$0 = \bar{Z}^2 - \frac{(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}\frac{1}{h'(\bar{A})}\bar{Z} - \frac{1}{[h'(\bar{A})]^2}\frac{-\beta g[h'(\bar{A})]^2\bar{\pi}_{SS} - (1-\beta g^2)h'(\bar{A})\bar{\pi}_{AS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}.$$

From the quadratic formula, the solution is

$$\bar{Z} = \left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} \pm \sqrt{discrim}\right] \left[2h'(\bar{A})\left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right)\right]^{-1},$$

where the discriminant is

$$discrim = \left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right)^2 + 4\left(-\beta g[h'(\bar{A})]^2\bar{\pi}_{SS} - (1-\beta g^2)h'(\bar{A})\bar{\pi}_{AS}\right)\left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]\right). \tag{A-11}$$

The proof of Lemma 2 will show that (5) implies that *discrim* is positive.

In order to analyze stability, linearize the difference equations. Substituting χ_t into Z_t, we find:

$$Z_t = \left[\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}\right] \left[\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g h'(\bar{A})\frac{\bar{\pi}_{AS}}{h'(\bar{A})} - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A})\left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}\right]^{-1}.$$

Linearizing and evaluating at the steady state:

$$\frac{\partial Z_t}{\partial Z_{t+1}}\bigg|_{\bar{Z}} = \left[2\beta g h'(\bar{A})\bar{\pi}_{AS} + (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} \pm \sqrt{discrim}\right]$$
$$\left[2\beta g h'(\bar{A})\bar{\pi}_{AS} + (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - \left(\pm\sqrt{discrim}\right)\right]^{-1}. \tag{A-12}$$

The terms outside the square root are positive if

$$-2\beta g h'(\bar{A})\bar{\pi}_{AS} < (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}. \tag{A-13}$$

The following lemma establishes that those terms are in fact positive:

Lemma 1. *Inequality* (1) *implies inequality* (A-13).

Proof. By the inequality of arithmetic and geometric means, inequality (1) implies

$$-h'(A_t)\pi_{AS} < \frac{1}{2}\left(-\pi_{AA} + \frac{h''(A_t)}{h'(A_t)}\pi_A\right) - \frac{1}{2}[h'(A_t)]^2\pi_{SS}.$$

Multiplying both sides by β and using inequality (2) and $1+\beta g^2 > \beta$, this inequality implies

$$-\beta h'(A_t)\pi_{AS} < \frac{1}{2}(1+\beta g^2)\left(-\pi_{AA} + \frac{h''(A_t)}{h'(A_t)}\pi_A\right) - \frac{1}{2}\beta[h'(A_t)]^2\pi_{SS}.$$

Using $g < 1$, this last inequality in turn implies inequality (A-13).

□

Because the terms outside the square root in (A-12) are positive, the numerator and denominator are both larger when the square root is added rather than subtracted. The stable steady state (with eigenvalue < 1 in magnitude) is therefore the one with a negative sign in the numerator of (A-12). The steady state of interest is therefore

$$\bar{Z} = \left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - \sqrt{discrim}\right]$$
$$\left[2h'(\bar{A})\left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right)\right]^{-1}. \tag{A-14}$$

Substituting into $\bar{\chi}$, we find:

$$\bar{\chi} = \frac{1}{2h'(\bar{A})}\left[(1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2\beta g h'(\bar{A})\bar{\pi}_{AS} + \sqrt{discrim}\right]. \tag{A-15}$$

From Lemma 1 and inequality (1), $h'(\bar{A})\,\bar{\chi} > 0$.

Now return to the case in which $\zeta = 0$ from some time t onward. We have derived an expression for $\tilde{A}_t$ as $T \to \infty$. Using this,

$$\tilde{A}_{t+3} = \bar{A} + \bar{Z}(S_{t+2} - \bar{S}).$$

At time $t+2$, the relevant Euler equation is:

$$0 = \frac{\pi_A(f_{2,t}, \tilde{A}_{t+2}, S_{t+2}; K)}{h'(\tilde{A}_{t+2})} + \beta \pi_S(C, \tilde{A}_{t+3}, S_{t+3}; K) + \beta g \, \frac{-\pi_A(C, \tilde{A}_{t+3}, S_{t+3}; K)}{h'(\tilde{A}_{t+3})},$$

where we recognize that $w_{t+2} = f_{2,t}$. A first-order approximation to $\tilde{A}_{t+2}$ around $S_{t+2} = \bar{S}$ and $f_{2,t} = C$ is exact when either Assumption 1 or 2 holds and $(S_{t+2} - \bar{S})^2$ is small. We thereby obtain

$$\tilde{A}_{t+2} = \bar{A} + \bar{Z}(S_{t+2} - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(f_{2,t} - C).$$

If $(S_{t+1} - \bar{S})^2$ is small and either Assumption 1 or 2 holds, then approximating A_{t+1} around $S_{t+1} = \bar{S}$, $w_{t+1} = f_{1,t} = C$, $f_{1,t+1} = f_{2,t} = C$, and $\zeta = 0$ in a version of the stochastic Euler equation (A-10) advanced by one timestep yields:

$$A_{t+1} = \bar{A} + \bar{Z}(S_{t+1} - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C),$$

where

$$\beta\Gamma \triangleq \beta h'(\bar{A})\bar{\pi}_{wS} - \beta g \bar{\pi}_{wA} + \beta \overbrace{\left[h'(\bar{A})\bar{\pi}_{AS} + g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right]}^{\triangleq\Psi} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}. \tag{A-16}$$

If $(S_t - \bar{S})^2$ is small and either Assumption 1 or 2 holds, then approximating A_t around $S_t = \bar{S}$, $w_t = C$, $f_{1,t} = C$, $f_{2,t} = C$, and $\zeta = 0$ in the stochastic Euler equation (A-10) yields:

$$A_t = \bar{A} + \bar{Z}(S_t - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_t - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C). \tag{A-17}$$

Throughout, the terms with ζ drop out due to the expectation operator in the stochastic Euler equation, and the terms with ζ^2 drop out due to the assumptions.

E.6 Evolution of expected actions and states

For $t \geq 2$,

$$E_0[A_t] = \bar{A} + \bar{Z}(E_0[S_t] - \bar{S}).$$

Approximate S_t around $A_{t-1} = \bar{A}$ and $S_{t-1} = \bar{S}$:

$$S_t \approx \bar{S} + h'(\bar{A})(A_{t-1} - \bar{A}) + g(S_{t-1} - \bar{S}).$$

We then have:

$$E_0[A_t] = \bar{A} + \bar{Z}h'(\bar{A})(E_0[A_{t-1}] - \bar{A}) + \bar{Z}g\left(E_0[S_{t-1}] - \bar{S}]\right).$$

Repeatedly substituting, we find:

$$E_0[A_t] = \bar{A} + [\bar{Z}h'(\bar{A}) + g]^{x-1}\left[\bar{Z}h'(\bar{A})(E_0[A_{t-x}] - \bar{A}) + \bar{Z}g(E_0[S_{t-x}] - \bar{S}])\right]$$

for $x \in \{1, ..., t-1\}$. Analogously,

$$E_0[S_t] = \bar{S} + [\bar{Z}h'(\bar{A}) + g]^{x-1}\left[h'(\bar{A})(E_0[A_{t-x}] - \bar{A}) + g(E_0[S_{t-x}] - \bar{S}])\right].$$

We have geometric series. The following lemma establishes that the common ratio is less than 1 in magnitude.

Lemma 2. (5) *implies* $|\bar{Z}h'(\bar{A}) + g| < 1$.

Proof. Observe that:

$$\bar{Z}h'(\bar{A}) + g = \frac{\overbrace{2g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) + 2h'(\bar{A})\bar{\pi}_{AS}}^{=2\Psi}}{(1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2\beta g h'(\bar{A})\bar{\pi}_{AS} + \sqrt{discrim}}, \tag{A-18}$$

where the numerator is equal to 2Ψ by (A-16). Recalling that inequality (1) implies inequality (A-13) (Lemma 1), the denominator is clearly positive. Rewrite (A-18) as:

$$\begin{aligned}\bar{Z}h'(\bar{A}) + g = &\left[g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) + h'(\bar{A})\bar{\pi}_{AS}\right] \\ &\Bigg[\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + \beta g h'(\bar{A})\bar{\pi}_{AS} \\ &\quad + \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right)\Bigg]^{-1}.\end{aligned} \tag{A-19}$$

We desire to show $\bar{Z}h'(\bar{A}) + g < 1$ if $\bar{Z}h'(\bar{A}) + g > 0$ and to show $\bar{Z}h'(\bar{A}) + g > -1$ if $\bar{Z}h'(\bar{A}) + g < 0$.

First consider $\bar{Z}h'(\bar{A}) + g > 0$. The first line of the denominator in (A-19) is positive and is larger than the numerator. The second line in the denominator is positive if and only if

$$\left\{-\beta g[h'(\bar{A})]^2\bar{\pi}_{SS} - (1-\beta g^2)h'(\bar{A})\bar{\pi}_{AS}\right\}\left\{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]\right\} > 0.$$

The expression contained in the second curly braces is positive because it is proportional to $\bar{Z}h'(\bar{A}) + g$. The expression contained in the first curly braces is positive if $h'(\bar{A})\bar{\pi}_{AS} \leq 0$. In this case, the inequality does hold and the second line of the denominator reinforces the first. So $\bar{Z}h'(\bar{A}) + g < 1$ if $\bar{Z}h'(\bar{A}) + g > 0$ and $h'(\bar{A})\bar{\pi}_{AS} \leq 0$.

If, instead, $\bar{Z}h'(\bar{A}) + g > 0$ with $h'(\bar{A})\bar{\pi}_{AS} > 0$, the second line of the denominator in (A-19) can be negative if $h'(\bar{A})\bar{\pi}_{AS}$ is sufficiently large. So we seek the largest value of $h'(\bar{A})\bar{\pi}_{AS}$ compatible with $\bar{Z}h'(\bar{A}) + g \leq 1$. Rearranging the inequality $\bar{Z}h'(\bar{A}) + g < 1$, we find:[52]

$$\begin{aligned}
1 >& \bar{Z}h'(\bar{A}) + g \\
\Leftrightarrow& \sqrt{discrim} > [2g - 1 - \beta g^2]\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) + \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2(1-\beta g)h'(\bar{A})\bar{\pi}_{AS}.
\end{aligned}$$

The right-hand side is positive in the region of interest, around where the inequality binds. Squaring both sides, this inequality becomes:

$$\begin{aligned}
0 <& g(1-g)(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2 - \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS} \\
&- \beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS} + [1 - 2g(1+\beta) + 3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] \\
&- [1 + \beta - 2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2. \qquad \text{(A-20)}
\end{aligned}$$

This is a quadratic in $h'(\bar{A})\bar{\pi}_{AS}$. It opens down. So the acceptable values of $h'(\bar{A})\bar{\pi}_{AS}$ will be in an intermediate range (if they exist). We already saw that the inequality must hold for small positive values of $h'(\bar{A})\bar{\pi}_{AS}$, so it should be the case that any roots are on either side of zero with the y-intercept strictly positive (as is easy to verify).[53] So $\bar{Z}h'(\bar{A}) + g < 1$ only if $h'(\bar{A})\bar{\pi}_{AS}$ is less than the positive root. Observe that the product of the constant and the quadratic coefficient is negative. Therefore,

[52]Doing so, it is easy to see that $discrim > 0$ if $\bar{Z}h'(\bar{A}) + g < 1$ which validates one half of an earlier claim (i.e., only for the case with $\bar{Z}h'(\bar{A}) + g > 0$) once we establish that $\bar{Z}h'(\bar{A}) + g < 1$.

[53]We also saw that the inequality must hold for negative values of $h'(\bar{A})\bar{\pi}_{AS}$, so readers may be confused by the fact that there is a negative root as well. But observe that sufficiently negative $h'(\bar{A})\bar{\pi}_{AS}$ is incompatible with $\bar{Z}h'(\bar{A}) + g > 0$.

from the quadratic formula, inequality (A-20) holds if

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1-2g(1+\beta)+3\beta g^2]\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]-\beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta-2\beta g}.$$

Indeed, this holds by (5). Therefore $\bar{Z}h'(\bar{A})+g<1$ if $\bar{Z}h'(\bar{A})+g>0$ and $h'(\bar{A})\bar{\pi}_{AS}>0$.

Finally, consider the case with $\bar{Z}h'(\bar{A})+g<0$. It must be true that $h'(\bar{A})\bar{\pi}_{AS}<0$. Rearranging the inequality $\bar{Z}h'(\bar{A})+g>-1$, we find:[54]

$$\begin{aligned} 1 >& -\left[\bar{Z}h'(\bar{A})+g\right] \\ \Leftrightarrow \sqrt{discrim} >& [-2g-1-\beta g^2]\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)+\beta[h'(\bar{A})]^2\bar{\pi}_{SS}+2(-1-\beta g)h'(\bar{A})\bar{\pi}_{AS}. \end{aligned}$$

The right-hand side must be positive in the region where $h'(\bar{A})\bar{\pi}_{AS}$ is sufficiently large in magnitude to make this inequality bind. Squaring both sides, this becomes:

$$\begin{aligned} 0 <& -g(1+g)(1+\beta g)\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2+\beta g\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS} \\ &+\beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS}+[-1-2g(1+\beta)-3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] \\ &-[1+\beta+2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2. \end{aligned} \tag{A-21}$$

This quadratic opens down. The y-intercept is strictly negative. The derivative at the y-intercept is:

$$\beta[h'(\bar{A})]^2\bar{\pi}_{SS}+[-1-2g(1+\beta)-3\beta g^2]\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]<0.$$

So both roots are negative. The root that is closer to zero does not bind the inequality of ultimate interest. (Indeed, $\bar{Z}h'(\bar{A})+g$ is not even negative for $h'(\bar{A})\bar{\pi}_{AS}$ close to 0.) Observe that the product of the constant and the quadratic coefficient is positive. Therefore, from the quadratic formula, inequality (A-21) holds if

$$h'(\bar{A})\bar{\pi}_{AS} > \frac{[-1-2g(1+\beta)-3\beta g^2]\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]+\beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta+2\beta g}.$$

Indeed, this holds by (5). Therefore $\bar{Z}h'(\bar{A})+g>-1$ if $\bar{Z}h'(\bar{A})+g<0$.

□

We therefore have, for $(S_0-\bar{S})^2$ not too large and under either Assumption 1 or 2,

$$\lim_{t\to\infty} E_0[A_t]=\bar{A} \text{ and } \lim_{t\to\infty} E_0[S_t]=\bar{S}.$$

[54]Doing so, it is easy to see that $discrim>0$ if $\bar{Z}h'(\bar{A})+g>-1$ which validates the remaining half of an earlier claim (i.e., now for the case with $\bar{Z}h'(\bar{A})+g<0$) once we establish that $\bar{Z}h'(\bar{A})+g>-1$.

E.7 Deriving equation (8)

Expand π_t around $w_t = C$, $A_t = \bar{A}$, and $S_t = \bar{S}$:

$$\begin{aligned}\pi_t =&\bar{\pi} + \bar{\pi}_w(w_t - C) + \bar{\pi}_A(A_t - \bar{A}) + \bar{\pi}_S(S_t - \bar{S})\\ &+ \frac{1}{2}\bar{\pi}_{ww}(w_t - C)^2 + \frac{1}{2}\bar{\pi}_{AA}(A_t - \bar{A})^2 + \frac{1}{2}\bar{\pi}_{SS}(S_t - \bar{S})^2\\ &+ \bar{\pi}_{wA}(w_t - C)(A_t - \bar{A}) + \bar{\pi}_{wS}(w_t - C)(S_t - \bar{S}) + \bar{\pi}_{AS}(A_t - \bar{A})(S_t - \bar{S}),\end{aligned} \tag{A-22}$$

where higher order terms vanish under either Assumption 1 or Assumption 2. Appendix E.6 showed that

$$\lim_{t\to\infty} E_0[A_t] = \bar{A} \text{ and } \lim_{t\to\infty} E_0[S_t] = \bar{S}$$

if $(S_0 - \bar{S})^2$ is not too large and either Assumption 1 or 2 holds. Using these and $E_0[w_t] = C$ for $t > 1$, we find:

$$\begin{aligned}\lim_{t\to\infty} E_0[\pi_t] =&\bar{\pi} + \frac{1}{2}\bar{\pi}_{ww} trace(\Sigma)\zeta^2 + \frac{1}{2}\bar{\pi}_{AA}E_0[(A_t - \bar{A})^2] + \frac{1}{2}\bar{\pi}_{SS}E_0[(S_t - \bar{S})^2]\\ &+ \bar{\pi}_{wA}E_0[(w_t - C)(A_t - \bar{A})] + \bar{\pi}_{wS}E_0[(w_t - C)(S_t - \bar{S})] + \bar{\pi}_{AS}E_0[(A_t - \bar{A})(S_t - \bar{S})].\end{aligned}$$

Differentiating and using either Assumption 1 or Assumption 2 again, we find

$$\lim_{t\to\infty} \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} =\bar{\pi}_w + \bar{\pi}_A\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} + \bar{\pi}_S\frac{\mathrm{d}\bar{S}}{\mathrm{d}C} + \bar{\pi}_K\frac{\mathrm{d}K}{\mathrm{d}C}.$$

Long-run payoffs under expected weather draws are $\bar{\pi}$. K is chosen such that $\bar{\pi}_K = 0$. From equation (A-7),

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}C} = \frac{h'(\bar{A})}{1-g}\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

Therefore

$$\lim_{t\to\infty} \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} =\bar{\pi}_w + \left[\bar{\pi}_A + \bar{\pi}_S\frac{h'(\bar{A})}{1-g}\right]\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

E.8 Deriving equation (11)

Implicitly differentiating equation (A-8), we have:

$$\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} =&\frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS}}{-(1-\beta g)\bar{\pi}_{AA} - \beta h''(\bar{A})\bar{\pi}_S - \beta h'(\bar{A})\frac{h'(\bar{A})}{1-g}\bar{\pi}_{SS} - \frac{1-\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}}\\ &+ \frac{(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}}{-(1-\beta g)\bar{\pi}_{AA} - \beta h''(\bar{A})\bar{\pi}_S - \beta h'(\bar{A})\frac{h'(\bar{A})}{1-g}\bar{\pi}_{SS} - \frac{1-\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}}\frac{\mathrm{d}K}{\mathrm{d}C}.\end{aligned}$$

Substitute for $\beta\bar{\pi}_S$ from equation (A-8):

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} = \frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS}}{(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \frac{1+\beta-2\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS}} + \frac{(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}}{(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \frac{1+\beta-2\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS}}\frac{\mathrm{d}K}{\mathrm{d}C}. \tag{A-23}$$

The denominator is strictly positive if and only if inequality (A-9) holds, which we saw indeeds hold when (5) holds. Therefore

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} \propto (1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + [(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}]\frac{\mathrm{d}K}{\mathrm{d}C}.$$

We have established the result we sought. For later use, note that if $(S_0 - \bar{S})^2$ is not too large and either Assumption 1 or Assumption 2 holds, then, from equation (10),

$$\lim_{t\to\infty}\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} = \bar{\pi}_w - \frac{1-\beta}{\beta}\bar{\pi}_A\frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + [(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}]\frac{\mathrm{d}K}{\mathrm{d}C}}{D}, \tag{A-24}$$

where

$$D \triangleq (1-g)(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - (1+\beta-2\beta g)h'(\bar{A})\bar{\pi}_{AS} \tag{A-25}$$

and $D > 0$ from (5). And observe that, from equations (A-15) and (A-25),

$$h'(\bar{A})\bar{\chi} = D + [1+\beta(1-g)]\left\{g\left[-\bar{\pi}_{AA} - h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] + h'(\bar{A})\bar{\pi}_{AS}\right\} + \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right). \tag{A-26}$$

E.9 Proof of Proposition 2

Expanding S_t around $\bar{A}$ and $\bar{S}$, we have, from Taylor's theorem,

$$S_t = \bar{S} + h'(\bar{A})(A_{t-1} - \bar{A}) + g(S_{t-1} - \bar{S}) + higherorderterms1,$$

where $higherorderterms1$ is a linear function of terms with $(A_{t-1} - \bar{A})^{\alpha_1}(S_{t-1} - \bar{S})^{\alpha_2}$ for $\alpha_1, \alpha_2 \in \mathbb{Z}_+$ and $\alpha_1 + \alpha_2 > 1$. Substituting for S_t then S_{t-1} and so on, equation (A-22) becomes:

$$\pi_t = \bar{\pi} + \bar{\pi}_w(w_t - C) + \bar{\pi}_A(A_t - \bar{A}) + \bar{\pi}_S h'(\bar{A}) \sum_{i=0}^{\infty} g^i (A_{t-1-i} - \bar{A}) + higherorderterms2, \tag{A-27}$$

where $higherorderterms2$ is a linear function of terms with $(w_t - C)^{\alpha_1}(A_{t-1-k} - \bar{A})^{\alpha_2}(S_{t-1-k} - \bar{S})^{\alpha_3}$ for $k \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_+$, and $\alpha_1 + \alpha_2 + \alpha_3 > 1$. If either Assumption 1 or 2 holds, then, substituting for S_t and then for A_{t_1} and S_{t-1} and so on, equation (A-17) becomes:

$$\begin{aligned} A_t =& \bar{A} + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_t - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C) \\ &+ \bar{Z}h'(\bar{A}) \sum_{i=0}^{\infty} [\bar{Z}h'(\bar{A}) + g]^i \Bigg[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_{t-1-i} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t-1-i} - C) \\ &\qquad + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t-1-i} - C) \Bigg] \\ &+ higherorderterms3, \end{aligned}$$

where $higherorderterms3$ is a linear function of terms with $(w_{t-k} - C)^{\alpha_1}(f_{1,t-k} - C)^{\alpha_2}(f_{2,t-k} - C)^{\alpha_3}$ for $k > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_+$, and $\alpha_1 + \alpha_2 + \alpha_3 > 1$. Using this and its analogues in (A-27), we find

$$\begin{aligned} \pi_t =& \bar{\pi} + \left[\bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right](w_t - C) + \bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t} - C) + \bar{\pi}_A \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C) \\ &+ \left\{\bar{\pi}_A \bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \\ &\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_{t-1} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t-1} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t-1} - C)\right] \\ &+ \sum_{i=2}^{\infty} \left\{\bar{\pi}_A \bar{Z}h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A}) g^{i-1} + \bar{\pi}_S \bar{Z}[h'(\bar{A})]^2 \sum_{j=1}^{i-1} [\bar{Z}h'(\bar{A}) + g]^{i-j-1} g^{j-1}\right\} \\ &\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_{t-i} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t-i} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t-i} - C)\right] \\ &+ higherorderterms4, \end{aligned} \tag{A-28}$$

where $higherorderterms4$ is a linear function of $higherorderterms2$ and $higherorderterms3$.

The vector of estimated coefficients is

$$\begin{bmatrix} \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\Lambda}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = E[X^\intercal X]^{-1} E[X^\intercal \boldsymbol{\pi}],$$

where $\hat{\boldsymbol{\alpha}}$ is a $J \times 1$ vector stacking the $\hat{\alpha}_j$; $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{\gamma}}$ are $I \times 1$ vectors stacking the $\hat{\Lambda}_i$, $\hat{\lambda}_i$, and $\hat{\gamma}_i$; $\boldsymbol{\pi}$ is a $JT \times 1$ vector with rows π_{jt}; and X is a $JT \times (J + 3I)$ matrix with the final $3I$ columns of each row being

$$\begin{bmatrix} w_{jt} & \dots & w_{j(t-I)} & f_{j1,t} & \dots & f_{j1,t-I} & f_{j2,t} & \dots & f_{j2,t-I} \end{bmatrix}.$$

By the Frisch-Waugh Theorem,

$$\begin{bmatrix} \hat{\boldsymbol{\Lambda}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = E[\tilde{X}^\intercal \tilde{X}]^{-1} E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}],$$

where $\tilde{X}$ is a $JT \times 3I$ matrix with rows

$$\begin{bmatrix} w_{jt} - C & \dots & w_{j(t-I)} - C & f_{j1,t} - C & \dots & f_{j1,t-I} - C & f_{j2,t} - C & \dots & f_{j2,t-I} - C \end{bmatrix}$$

and $\tilde{\boldsymbol{\pi}}$ is demeaned $\boldsymbol{\pi}$. Observe that:

$$E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}] = JT \begin{bmatrix} Cov[w_{jt} - C, \pi_{jt}] \\ \vdots \\ Cov[w_{j(t-I)} - C, \pi_{jt}] \\ Cov[f_{j1,t} - C, \pi_{jt}] \\ \vdots \\ Cov[f_{j1,t-I} - C, \pi_{jt}] \\ Cov[f_{j2,t} - C, \pi_{jt}] \\ \vdots \\ Cov[f_{j2,t-I} - C, \pi_{jt}] \end{bmatrix}.$$

Following the proof of Proposition 1, $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{\gamma}}$ are independent of $higherorderterms4$ if either Assumption 1 or Assumption 3 holds. From here, drop the j subscript to save on unnecessary notation.

Observe that $Cov[w_{t-k}, w_{t-k-j}] = Cov[w_{t-k}, f_{1,t-k-j}] = Cov[w_{t-k}, f_{2,t-k-j}] = 0$ for $j > 2$, that $Cov[f_{1,t-k}, w_{t-k-j}] = Cov[f_{1,t-k}, f_{1,t-k-j}] = Cov[f_{1,t-k}, f_{2,t-k-j}] = 0$ for $j > 1$, and that $Cov[f_{2,t-k}, w_{t-k-j}] = Cov[f_{2,t-k}, f_{1,t-k-j}] = Cov[f_{2,t-k}, f_{2,t-k-j}] = 0$ for $j > 0$. It is obvious from standard regression results on omitted variables bias (and verifiable through tedious algebra) that, for $i < I - 1$, the probability limits of $\hat{\Lambda}_i$, $\hat{\lambda}_i$, and $\hat{\gamma}_i$ are identical to the coefficients on, respectively, $w_{t-i} - C$, $f_{1,t-i} - C$, and $f_{2,t-i} - C$ in equation (A-28). We then find:

$$
\begin{aligned}
&\lim_{I\to\infty} \operatorname{plim} \sum_{i=0}^{I-2} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right] \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
&+ \sum_{i=2}^{\infty} \left\{\bar{\pi}_A \bar{Z} h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A}) g^{i-1} + \bar{\pi}_S \bar{Z}[h'(\bar{A})]^2 \sum_{j=1}^{i-1} [\bar{Z}h'(\bar{A}) + g]^{i-j-1} g^{j-1}\right\} \\
&\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right] \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
&+ \sum_{i=2}^{\infty} \left\{\bar{\pi}_A \bar{Z} h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A}) (\bar{Z}h'(\bar{A}) + g)^{i-1}\right\} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right],
\end{aligned}
$$

where we used Lemma 2 to establish that the common ratio is less than 1 in magnitude. Substituting for $\bar{\pi}_S$ from equation (A-8),

$$
\frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]} = -\frac{1}{\beta}\bar{\pi}_A \frac{1 - \beta[\bar{Z}h'(\bar{A}) + g]}{1 - [\bar{Z}h'(\bar{A}) + g]}.
$$

Then, using equations (A-15) and (A-18),

$$
\begin{aligned}
&\lim_{I\to\infty} \operatorname{plim} \sum_{i=0}^{I-2} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] \\
=&\bar{\pi}_w \\
&- \frac{1-\beta}{\beta}\bar{\pi}_A \left\{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right] \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right\} \\
&\left\{\frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_{AA}}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + \sqrt{discrim}\right]\right. \\
&\qquad \left. - g(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_{AA}}{h'(\bar{A})}\right) - (1-\beta g)h'(\bar{A})\bar{\pi}_{AS}\right\}^{-1} \\
=&\bar{\pi}_w \\
&- \frac{1-\beta}{\beta}\bar{\pi}_A \left\{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right] \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right\} \\
&\left\{D + \beta(1-g)\Psi + \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right]\right\}^{-1},
\end{aligned}
$$

where the last equality uses equation (A-25). Using equations (A-8), (A-23), and (A-24),

$$\lim_{I\to\infty} \operatorname{plim} \sum_{i=0}^{I-2} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] = \bar{\pi}_w + \omega \left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right] \left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right),$$

where

$$\Omega \triangleq \frac{\beta\Psi \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}}{D/(1-g)}, \tag{A-29}$$

$$\begin{aligned}\omega \triangleq D \Bigg\{ D & \\ &+ \beta(1-g)\Psi \\ &+ \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right]\Bigg\}^{-1},\end{aligned} \tag{A-30}$$

and, from equation (14),

$$\Psi \triangleq h'(\bar{A})\bar{\pi}_{AS} + g \underbrace{\left(-\bar{\pi}_{AA} + \frac{h''(\bar{A})}{h'(\bar{A})}\bar{\pi}_A\right)}_{>0 \text{ by } (2)}.$$

D, from equation (A-25), is positive if and only if inequality (A-9) holds, which we saw indeeds hold by (5). Observe that, from (A-23), the denominator of $\mathrm{d}\bar{A}/\,\mathrm{d}C$ is $D/(1-g)$.

Analyze ω by considering the divergence between the terms in curly braces in (A-30) and D. First, if $\beta\Psi = 0$, then the second line in curly braces is zero and, from equation (A-11), so is the third line in curly braces. Therefore $\omega = 1$ if $\beta\Psi = 0$.

Next, if $\beta\Psi < 0$, then the second line in curly braces is strictly negative. Further, $h'(\bar{A})\bar{\pi}_{AS}$ must be weakly negative. From equation (A-11), $\beta\Psi < 0$ and $h'(\bar{A})\bar{\pi}_{AS} \leq 0$ imply that the third line in curly braces is negative. Using (A-26), the denominator of ω is strictly greater than $h'(\bar{A})\bar{\chi}$ when $\Psi < 0$. From equation (A-15), Lemma 1, and inequality (1), $h'(\bar{A})\,\bar{\chi} > 0$. Therefore the denominator of ω is strictly positive when $\Psi < 0$. And because the combined terms in curly braces in (A-30) are strictly less than D, we have established that $\omega > 1$ if $\beta\Psi < 0$.

If $\beta\Psi > 0$, then the second line in curly braces in (A-30) is strictly positive. From equation (A-11), the third line in curly braces is positive if $\beta\Psi > 0$ and $h'(\bar{A})\bar{\pi}_{AS}$ is not too much greater than 0. In that case, $\omega < 1$.

Finally, consider $\beta\Psi > 0$ with $h'(\bar{A})\bar{\pi}_{AS}$ strictly positive and sufficiently large to make the third line in curly braces negative. Consider whether that line can be so negative as to overwhelm the positive second line in curly braces and make $\omega > 1$.

Those final two lines in curly braces are strictly positive with $h'(\bar{A})\bar{\pi}_{AS} > 0$ if and only if

$$\sqrt{discrim} > [1 + \beta g^2 - 2\beta g]\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - 2\beta(1-g)h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}.$$

Squaring both sides, this inequality holds if and only if

$$\begin{aligned}0 <& g(1-g)(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2 - \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS} \\ &- \beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS} + [1 - 2g(1+\beta) + 3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] \\ &- [1 + \beta - 2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2.\end{aligned}$$

This last inequality is identical to inequality (A-20), which we saw holds by (5). Therefore $\omega < 1$ if $\beta\Psi > 0$.

E.10 Proof of Corollary 3

First consider $I' > 1$. As described in the proof of Proposition 2, the probability limits of $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ are, for $i < I - 1$, identical to the coefficients on $w_{t-i} - C$ and $f_{t-i} - C$ in equation (A-28). Using Lemma 2 to establish that the common ratio is not equal to 1, we obtain:

$$\begin{aligned}\text{plim}\sum_{i=0}^{I'}\left[\hat{\Lambda}_i + \hat{\lambda}_i\right] =& \bar{\pi}_w + \bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right]\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\ &+ \sum_{i=2}^{I'}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A})(\bar{Z}h'(\bar{A}) + g)^{i-1}\right\}\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\ =& \bar{\pi}_w + \bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\ &+ \left[1 - [\bar{Z}h'(\bar{A}) + g]^{I'}\right]\frac{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]}\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].\end{aligned}$$

Substituting for $\bar{\pi}_S$ from equation (A-8),

$$\begin{aligned}&\bar{\pi}_A + \left[1 - [\bar{Z}h'(\bar{A}) + g]^{I'}\right]\frac{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]} \\ =& -\frac{1-\beta}{\beta}\bar{\pi}_A\frac{1}{1 - [\bar{Z}h'(\bar{A}) + g]}\left\{1 - [\bar{Z}h'(\bar{A}) + g]^{I'}\frac{1 - \beta[\bar{Z}h'(\bar{A}) + g]}{1-\beta}\right\}.\end{aligned}$$

For $I' = 1$, we have:

$$\text{plim}\sum_{i=0}^{1}\left[\hat{\Lambda}_i + \hat{\lambda}_i\right] = \bar{\pi}_w + \bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right]\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].$$

Substituting for $\bar{\pi}_S$ from equation (A-8) and rearranging, we find:

$$\bar{\pi}_A + [\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})] = -\frac{1-\beta}{\beta}\bar{\pi}_A \frac{1}{1-[\bar{Z}h'(\bar{A})+g]}\left[1-[\bar{Z}h'(\bar{A})+g]\frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right].$$

Using these results and following the analysis of Proposition 2, we have, for $I' \geq 1$,

$$\text{plim} \sum_{i=0}^{I'} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] = \bar{\pi}_w + \omega_{I'} \left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right] \left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right),$$

where

$$\omega_{I'} \triangleq \left\{1 - [\bar{Z}h'(\bar{A})+g]^{I'} \frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right\}\omega. \tag{A-31}$$

and where Ω and ω are as in Proposition 2. From equation (A-18), $\bar{Z}h'(\bar{A}) + g \propto \Psi$. Thus $\bar{Z}h'(\bar{A}) + g = 0$ if $g = \bar{\pi}_{AS} = 0$. In that case, $\omega_{I'} = \omega$ for all $I' \geq 1$. If $\Psi > 0$, then, using Lemma 2, the combined terms in curly braces in (A-31) are strictly positive, strictly less than 1, and strictly increasing in I'. In that case, following the analysis in Proposition 2, $\omega_{I'} \in (0, \omega)$ and $\omega_{I'}$ increases in I'. If $\Psi < 0$, then the combined terms in curly braces in (A-31) are strictly greater than 1 for I' odd. The statement of the corollary follows from the analysis of Proposition 2.

E.11 Proof of Corollary 4

First, observe that

$$\lim_{g,\bar{\pi}_{AS}\to 0} \Omega = 0$$

because

$$\lim_{g,\bar{\pi}_{AS}\to 0} \Psi = 0.$$

It is also obvious that

$$\lim_{\beta\to 0} \Omega = 0$$

Second, from equation (A-11),

$$\lim_{\beta\to 0} discrim = \lim_{g,\bar{\pi}_{AS}\to 0} discrim = \left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2 \bar{\pi}_{SS}\right)^2.$$

Using (A-30), this implies

$$\lim_{\beta\to 0} \omega = \lim_{g,\bar{\pi}_{AS}\to 0} \omega = 1.$$

Third, observe that

$$\lim_{\beta\to 0} \hat{\lambda}_i = 0$$

because

$$\lim_{\beta\to 0} \beta\Gamma = 0.$$

Finally, recall that Corollary 3 established that $\omega_{I'} = \omega$ when $g = \bar{\pi}_{AS} = 0$.

E.12 Proof of Proposition 5

Following the proof of Proposition 2 and using equation (A-28), we have:

$$
\begin{aligned}
\text{plim}\,\hat{\Lambda}_0 =& \bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\Lambda}_1 =& \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\Lambda}_2 =& [\bar{Z} h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_0 =& \bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_1 =& \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_2 =& [\bar{Z} h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.
\end{aligned}
$$

Observe that

$$
\text{plim}\,\frac{\hat{\lambda}_0}{\hat{\lambda}_1} = \frac{\bar{\pi}_A}{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})},
$$

$$
\text{plim}\,\frac{\hat{\Lambda}_2}{\hat{\Lambda}_1} = \bar{Z} h'(\bar{A}) + g,
$$

and, using equation (A-18),

$$
\Psi \propto \bar{Z} h'(\bar{A}) + g.
$$

Substituting for $\mathrm{d}\bar{A}/\mathrm{d}C$ from Appendix E.8, equation (10) becomes:

$$
\begin{aligned}
\lim_{t\to\infty} \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} =& \overbrace{\bar{\pi}_w}^{\text{direct effects}} \overbrace{- \frac{1-\beta}{\beta}\bar{\pi}_A \frac{\bar{\pi}_{wA}}{D}}^{\text{ex-post adaptation}} \overbrace{- \frac{1-\beta}{\beta}\bar{\pi}_A \frac{\beta[h'(\bar{A})\bar{\pi}_{wS} - g\,\bar{\pi}_{wA}]}{D}}^{\text{ex-ante adaptation}}\\
&\underbrace{- \frac{1-\beta}{\beta}\frac{\bar{\pi}_A}{D}\left[(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}\right]\frac{\mathrm{d}K}{\mathrm{d}C}}_{\text{interactions with long-lived infrastructure}},
\end{aligned}
\tag{A-32}
$$

with $D > 0$ itself a function of cross-partials.[55]

Rearranging the foregoing results, we find:

$$
\bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}} = \text{plim}\,\hat{\lambda}_0, \qquad \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} = \text{plim}\,\hat{\Lambda}_1 \frac{\hat{\lambda}_0}{\hat{\lambda}_1}, \qquad \bar{\pi}_w = \text{plim}\left(\hat{\Lambda}_0 - \hat{\Lambda}_1 \frac{\hat{\lambda}_0}{\hat{\lambda}_1}\right).
$$

[55]See equations (A-24) and (A-25) in Appendix E.8. Note that D absorbs the $1-g$ in the denominator of (10).

Using these results and labeling pieces as in (A-32), we can calculate the overall effect of climate:

$$\begin{aligned}
&\operatorname{plim}\left(\hat{\Lambda}_0 - \hat{\Lambda}_1 \frac{\hat{\lambda}_0}{\hat{\lambda}_1} - \frac{1-\beta}{\beta}\left[\hat{\Lambda}_1 \frac{\hat{\lambda}_0}{\hat{\lambda}_1} + \hat{\lambda}_0\right]\right) \\
=&\bar{\pi}_w - \frac{1-\beta}{\beta}\bar{\pi}_A\left\{\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}}\right\} \\
=&\bar{\pi}_w - \frac{D}{h'(\bar{A})\bar{\chi}}\frac{1-\beta}{\beta}\bar{\pi}_A\left(\frac{\bar{\pi}_{wA}}{D} + \frac{\beta[h'(\bar{A})\bar{\pi}_{wS} - g\,\bar{\pi}_{wA}]}{D} + \frac{\Omega}{1-g}\right) \\
=&\bar{\pi}_w + \frac{D}{h'(\bar{A})\bar{\chi}}\left[\bar{\pi}_A + \bar{\pi}_S\frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right). \qquad \text{(A-33)}
\end{aligned}$$

The second line uses foregoing results to express the calculation in terms of model primitives. The third line substitutes for Γ. Substituting $\mathrm{d}\bar{A}/\,\mathrm{d}C$ and also $\bar{\pi}_S$ from the Euler equation (9), the final line indicates how close we get to the true effect of climate from (8).

Consider the bias Ω. Following the proof of Proposition 2 and using equation (A-28), we have:

$$\operatorname{plim}\hat{\gamma}_0 = \bar{\pi}_A\frac{\beta\,\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.$$

Therefore

$$\operatorname{plim}\frac{\hat{\gamma}_0}{\hat{\lambda}_0} = \frac{\beta\,\Psi}{h'(\bar{A})\bar{\chi}}.$$

From equation (A-29),

$$\bar{\pi}_A\frac{\Omega}{1-g} = \bar{\pi}_A\frac{h'(\bar{A})\bar{\chi}}{D}\frac{\hat{\gamma}_0}{\hat{\lambda}_0}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}.$$

Substituting from foregoing results for $\bar{\pi}_A\bar{\pi}_{wA}/[h'(\bar{A})\,\bar{\chi}]$, we find:

$$\frac{D}{h'(\bar{A})\bar{\chi}}\bar{\pi}_A\frac{\Omega}{1-g} = \operatorname{plim}\frac{\hat{\gamma}_0}{\hat{\lambda}_0}\hat{\Lambda}_1\frac{\hat{\lambda}_0}{\hat{\lambda}_1}.$$

Combining this with (A-33) and defining $\tilde{\omega} \triangleq D/[h'(\bar{A})\bar{\chi}]$ yields equation (17) in the proposition.

Using equation (A-26), we have:

$$\begin{aligned}
\frac{D}{h'(\bar{A})\bar{\chi}} =&D\Bigg\{D \\
&+ [1+\beta(1-g)]\Psi \\
&+ \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right)\Bigg\}^{-1}.
\end{aligned}$$

Comparing to equation (A-30), we here have a coefficient of $[1+\beta(1-g)]$ on Ψ instead of $\beta(1-g)$. The analysis of $\Psi \leq 0$ is as in the case of $\beta\Psi \leq 0$ from before, except now $\beta = 0$ does not bring the second line in curly braces to zero. For $\Psi > 0$, note that it is now even harder for the third line in curly braces to overwhelm the second line, so if that could not happen for ω with $\beta\Psi > 0$, then it cannot happen here either for $\Psi > 0$.

E.13 Proof of Proposition 6

Following the proof of Proposition 2 and using equation (A-28), we have:

$$\begin{aligned}
\operatorname{plim}\hat{\phi}_0 =& \bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\operatorname{plim}\hat{\phi}_1 =& \left\{ \bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A}) \right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\operatorname{plim}\hat{\phi}_2 =& [\bar{Z} h'(\bar{A}) + g] \left\{ \bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A}) \right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}.
\end{aligned}$$

Substitute for $\bar{\pi}_S h'(\bar{A})$ from (A-8):

$$\operatorname{plim}\hat{\phi}_1 = \left\{ \bar{Z} h'(\bar{A}) + g - \frac{1}{\beta} \right\} \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}$$

Observe that

$$\operatorname{plim}\frac{\hat{\phi}_2}{\hat{\phi}_1} = \bar{Z} h'(\bar{A}) + g.$$

And using equation (A-18),

$$\Psi \propto \bar{Z} h'(\bar{A}) + g.$$

Rearranging the foregoing results, we have:

$$\bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} = \operatorname{plim}\frac{\hat{\phi}_1}{\frac{\hat{\phi}_2}{\hat{\phi}_1} - \frac{1}{\beta}}, \qquad \bar{\pi}_w = \operatorname{plim}\left(\hat{\phi}_0 - \frac{\hat{\phi}_1}{\frac{\hat{\phi}_2}{\hat{\phi}_1} - \frac{1}{\beta}} \right).$$

The proposition follows the proof of Proposition 5 from here.

E.14 Proof of Proposition A-1

Let there be N aggregated timesteps in total. We seek

$$\begin{bmatrix} \hat{\hat{\Lambda}} \\ \hat{\hat{\lambda}} \end{bmatrix} = E[\tilde{X}^\intercal \tilde{X}]^{-1} E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}], \tag{A-34}$$

where, guided by previous proofs, $\tilde{X}$ is a $JN \times 2$ matrix with rows

$$\begin{bmatrix} \breve{w}_{js} - C & \breve{f}_{j1,s} - C \end{bmatrix}$$

and $\tilde{\breve{\boldsymbol{\pi}}}$ is demeaned $\breve{\boldsymbol{\pi}}$. Observe that:

$$E[\tilde{X}^\intercal \tilde{\breve{\boldsymbol{\pi}}}] = JN \begin{bmatrix} Cov[\breve{w}_{js} - C, \breve{\pi}_{js}] \\ Cov[\breve{f}_{j1,s} - C, \breve{\pi}_{js}] \end{bmatrix}.$$

From here, drop the j subscript to avoid excess notation. After applying the Frisch-Waugh Theorem to partial out the effects of forecasts, correlations between payoffs and weather within a timestep are controlled by the coefficients on weather in equation (A-28). The exception is that variation in w_s also picks up the effect of $f_{1,s-1}$ because the latter variable is missing from $\breve{f}_{1,s}$. We then have:

$$\operatorname{plim} \hat{\breve{\Lambda}} = \operatorname{plim} \sum_{i=0}^{\Delta-1} \frac{\Delta - i}{\Delta} \hat{\Lambda}_i + \operatorname{plim} \frac{1}{\Delta} \sum_{i=0}^{\Delta-1} \frac{\Sigma_{22}}{\Sigma_{11} + \Sigma_{22}} \hat{\lambda}_{i+1}.$$

Analogously, we find:

$$\operatorname{plim} \hat{\breve{\lambda}} = \operatorname{plim} \sum_{i=0}^{\Delta-1} \frac{\Delta - i}{\Delta} \hat{\lambda}_i.$$

Therefore:

$$\begin{aligned} \operatorname{plim} \left(\hat{\breve{\Lambda}} + \hat{\breve{\lambda}} \right) &= \operatorname{plim} \left(\sum_{i=0}^{\Delta-1} \frac{\Delta - i}{\Delta} \left[\hat{\Lambda}_i + \hat{\lambda}_i \right] + \frac{1}{\Delta} \sum_{i=0}^{\Delta-1} \frac{\Sigma_{22}}{\Sigma_{11} + \Sigma_{22}} \hat{\lambda}_{i+1} \right) \\ &= \operatorname{plim} \left(\hat{\Lambda}_0 + \hat{\lambda}_0 + \sum_{i=1}^{\Delta-1} \frac{\Delta - i}{\Delta} \hat{\Lambda}_i + \sum_{i=1}^{\Delta-1} \frac{\Delta - i + \frac{\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}}{\Delta} \hat{\lambda}_i + \frac{\frac{\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}}{\Delta} \hat{\lambda}_\Delta \right). \end{aligned} \tag{A-35}$$

The coefficients on $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ are each $\in [0,1]$. In the proof of Corollary 3 (Appendix E.10), we established that

$$\begin{aligned} \operatorname{plim} \sum_{i=0}^{I'} \left[\hat{\Lambda}_i + \hat{\lambda}_i \right] =& \bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta \Gamma}{h'(\bar{A}) \bar{\chi}} \right] \\ &+ \left[1 - [\bar{Z} h'(\bar{A}) + g]^{I'} \right] \frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z} h'(\bar{A}) + g]} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta \Gamma}{h'(\bar{A}) \bar{\chi}} \right] \\ =& \bar{\pi}_w - \frac{1 - \beta}{\beta} \bar{\pi}_A \frac{1}{1 - [\bar{Z} h'(\bar{A}) + g]} \left\{ 1 - [\bar{Z} h'(\bar{A}) + g]^{I'} \frac{1 - \beta [\bar{Z} h'(\bar{A}) + g]}{1 - \beta} \right\} \\ & \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta \Gamma}{h'(\bar{A}) \bar{\chi}} \right]. \end{aligned}$$

In equation (A-35), each of the coefficients $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ is weighted by a fraction. Using $I' = \Delta$ in the previous expression, there exist $x_1 \in (0,1)$ and $x_2 \in (0,1)$ such that, for $\Psi > 0$,

$$\text{plim}\left(\hat{\mathring{\Lambda}} + \hat{\mathring{\lambda}}\right) = \bar{\pi}_w - \frac{1-\beta}{\beta}\bar{\pi}_A \frac{1}{1-[\bar{Z}h'(\bar{A})+g]}\left\{1-[\bar{Z}h'(\bar{A})+g]^{I'}\frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right\} \left[x_1 \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + x_2 \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].$$

Following the proof of Corollary E.10, there exists $x \in [x_1, x_2]$ (or $x \in [x_2, x_1]$ if $x_2 < x_1$) such that, for $\Psi > 0$,

$$\text{plim}\left(\hat{\mathring{\Lambda}} + \hat{\mathring{\lambda}}\right) = \bar{\pi}_w + x\,\omega_\Delta \left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right).$$

The statement of the proposition follows from defining $\breve{\omega} \triangleq x\,\omega_\Delta$.

E.15 Proof of Proposition A-2

The vector of estimated coefficients is

$$\begin{bmatrix}\hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\Phi}}\end{bmatrix} = E[X^\intercal X]^{-1}E[X^\intercal \boldsymbol{\pi}],$$

where $\hat{\boldsymbol{\alpha}}$ is a $J \times 1$ vector stacking the $\hat{\alpha}_j$, $\hat{\boldsymbol{\Phi}}$ is a 5×1 vector stacking the $\hat{\Phi}_i$, $\boldsymbol{\pi}$ is a $JT \times 1$ vector with rows π_{jt}, and X is a $JT \times (J+5)$ matrix with the final 5 columns of each row being

$$\begin{bmatrix} w_{j(t+2)} & w_{j(t+1)} & w_{jt} & w_{j(t-1)} & w_{j(t-2)} \end{bmatrix}.$$

By the Frisch-Waugh Theorem,

$$\hat{\boldsymbol{\Phi}} = E[\tilde{X}^\intercal \tilde{X}]^{-1}E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}],$$

where $\tilde{X}$ is a $JT \times 5$ matrix with rows

$$\begin{bmatrix} w_{j(t+2)} - C & w_{j(t+1)} - C & w_{jt} - C & w_{j(t-1)} - C & w_{j(t-2)} - C \end{bmatrix}$$

and $\tilde{\boldsymbol{\pi}}$ is demeaned $\boldsymbol{\pi}$. Observe that:

$$E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}] = JT \begin{bmatrix} Cov[w_{j(t+2)} - C, \pi_{jt}] \\ Cov[w_{j(t+1)} - C, \pi_{jt}] \\ Cov[w_{jt} - C, \pi_{jt}] \\ Cov[w_{j(t-1)} - C, \pi_{jt}] \\ Cov[w_{j(t-2)} - C, \pi_{jt}] \end{bmatrix}.$$

From here, drop the j subscript to save on unnecessary notation.

Following the proof of Proposition 1, using equation (A-28), and using Σ being diagonal, we find:

$$
\begin{aligned}
\frac{1}{\zeta^2}Cov[w_{t+2},\pi_t] =& \Sigma_{33}\bar{\pi}_A \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\frac{1}{\zeta^2}Cov[w_{t+1},\pi_t] =& \Sigma_{22}\bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}} + \Sigma_{33}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\frac{1}{\zeta^2}Cov[w_t,\pi_t] =& trace(\Sigma)\left[\bar{\pi}_w + \bar{\pi}_A\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right] + \Sigma_{22}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \Sigma_{33}\left\{\bar{Z}h'(\bar{A}) + g\right\}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\frac{1}{\zeta^2}Cov[w_{t-1},\pi_t] =& trace(\Sigma)\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\\
&+ \Sigma_{22}\left\{\bar{Z}h'(\bar{A}) + g\right\}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \Sigma_{33}\left\{\bar{Z}h'(\bar{A}) + g\right\}^2\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\frac{1}{\zeta^2}Cov[w_{t-2},\pi_t] =& trace(\Sigma)\left\{\bar{Z}h'(\bar{A}) + g\right\}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\\
&+ \Sigma_{22}\left\{\bar{Z}h'(\bar{A}) + g\right\}^2\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \Sigma_{33}\left\{\bar{Z}h'(\bar{A}) + g\right\}^3\left\{\bar{\pi}_A\bar{Z}h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.
\end{aligned}
$$

Σ diagonal implies that $E[\tilde{X}^\intercal\tilde{X}]^{-1}$ is a 5×5 diagonal matrix with $1/[\zeta^2 trace(\Sigma)]$ on the diagonal. Therefore,

$$
\begin{aligned}
\operatorname{plim}\hat{\Phi}_{-2} =& \frac{\Sigma_{33}}{trace(\Sigma)}\bar{\pi}_A \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\operatorname{plim}\hat{\Phi}_{-1} =& \frac{\Sigma_{22}}{trace(\Sigma)}\bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}} + \frac{\Sigma_{33}}{trace(\Sigma)}\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\operatorname{plim}\hat{\Phi}_{0} =& \bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\Sigma_{22}}{trace(\Sigma)}\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \frac{\Sigma_{33}}{trace(\Sigma)}[\bar{Z}h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\operatorname{plim}\hat{\Phi}_{1} =& \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\Sigma_{22}}{trace(\Sigma)}[\bar{Z}h' + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \frac{\Sigma_{33}}{trace(\Sigma)}[\bar{Z}h'(\bar{A}) + g]^2\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\operatorname{plim}\hat{\Phi}_{2} =& [\bar{Z}h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\\
&+ \frac{\Sigma_{22}}{trace(\Sigma)}[\bar{Z}h'(\bar{A}) + g]^2\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&+ \frac{\Sigma_{33}}{trace(\Sigma)}[\bar{Z}h'(\bar{A}) + g]^3\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\}\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.
\end{aligned}
$$

Using equation (A-8),

$$\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A}) = \bar{\pi}_A \left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right].$$

And observe that

$$\operatorname{plim}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} = \bar{Z}h'(\bar{A}) + g.$$

Then:

$$\begin{aligned}
\frac{\Sigma_{22}}{trace(\Sigma)}\bar{\pi}_A\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}} &= \operatorname{plim}\left(\hat{\Phi}_{-1} - \frac{\Sigma_{33}}{trace(\Sigma)}\bar{\pi}_A\left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right]\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right)\\
&= \operatorname{plim}\left(\hat{\Phi}_{-1} - \left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right]\hat{\Phi}_{-2}\right),\\
\bar{\pi}_A\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} &= \operatorname{plim}\frac{1}{\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}}\Bigg\{\hat{\Phi}_1 - \frac{\Sigma_{22}}{trace(\Sigma)}[\bar{Z}h' + g]\bar{\pi}_A\left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right]\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&\qquad - \frac{\Sigma_{33}}{trace(\Sigma)}[\bar{Z}h' + g]^2\bar{\pi}_A\left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right]\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Bigg\}\\
&= \operatorname{plim}\left(\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} + \left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right]\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \hat{\Phi}_{-2}\left(\frac{\hat{\Phi}_2}{\hat{\Phi}_1}\right)^2\right)\\
&= \operatorname{plim}\left(\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1}\right),\\
\bar{\pi}_w &= \operatorname{plim}\hat{\Phi}_0 - \bar{\pi}_A\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} - \frac{\Sigma_{22}}{trace(\Sigma)}\bar{\pi}_A\left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right]\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&\quad - \frac{\Sigma_{33}}{trace(\Sigma)}[\bar{Z}h'(\bar{A}) + g]\bar{\pi}_A\left[\bar{Z}h'(\bar{A}) + g - \frac{1}{\beta}\right]\frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\\
&= \operatorname{plim}\left(\hat{\Phi}_0 - \frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} + \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} + \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \hat{\Phi}_{-1}\left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right] + \left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right]^2\hat{\Phi}_{-2}\right)\\
&\quad - \operatorname{plim}\hat{\Phi}_{-2}\left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right]\frac{\hat{\Phi}_2}{\hat{\Phi}_1}\\
&= \operatorname{plim}\left(\hat{\Phi}_0 - \frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} + \frac{1}{\beta}\hat{\Phi}_{-1} + \frac{1}{\beta^2}\hat{\Phi}_{-2}\right).
\end{aligned}$$

Finally, using equation (A-18),

$$\Psi \propto \bar{Z}h'(\bar{A}) + g = \operatorname{plim}\frac{\hat{\Phi}_2}{\hat{\Phi}_1}.$$

We can use these model primitives to calculate the overall effect of climate:

$$\begin{aligned}
\text{plim}\Bigg(&\hat{\Phi}_0 - \frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} + \frac{1}{\beta}\hat{\Phi}_{-1} + \frac{1}{\beta^2}\hat{\Phi}_{-2} \\
&- \frac{1-\beta}{\beta}\left[\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} + \hat{\Phi}_{-1} - \left[\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\right]\hat{\Phi}_{-2}\right]\Bigg) \\
=&\bar{\pi}_w - \frac{1-\beta}{\beta}\left\{\bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\Sigma_{22}}{trace(\Sigma)}\bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}}\right\} \\
=&\bar{\pi}_w + \frac{D}{h'(\bar{A})\bar{\chi}}\left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right) - \left[\bar{\pi}_A + \bar{\pi}_S\frac{h'(\bar{A})}{1-g}\right]\frac{\Sigma_{11}+\Sigma_{33}}{trace(\Sigma)}\frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}}.
\end{aligned}$$

Now consider the calculation of Ω. From foregoing results,

$$\text{plim}\,\frac{\hat{\Phi}_{-2}}{\hat{\Phi}_{-1}} = \frac{\Sigma_{33}}{\Sigma_{22}}\frac{\beta\,\Psi}{h'(\bar{A})\bar{\chi}}.$$

From equation (A-29),

$$\bar{\pi}_A \frac{\Omega}{1-g} = \text{plim}\,\frac{\Sigma_{22}}{\Sigma_{33}}\bar{\pi}_A \frac{h'(\bar{A})\bar{\chi}}{D}\frac{\hat{\Phi}_{-2}}{\hat{\Phi}_{-1}}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}.$$

Using foregoing results to substitute for $\bar{\pi}_A\bar{\pi}_{wA}/[h'(\bar{A})\,\bar{\chi}]$, we find:

$$\frac{D}{h'(\bar{A})\bar{\chi}}\frac{\Sigma_{33}}{\Sigma_{22}}\bar{\pi}_A\frac{\Omega}{1-g} = \text{plim}\,\frac{\hat{\Phi}_{-2}}{\hat{\Phi}_{-1}}\left[\frac{\hat{\Phi}_1}{\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}} - \hat{\Phi}_{-1}\frac{\hat{\Phi}_2}{\hat{\Phi}_1} - \frac{1}{\beta}\hat{\Phi}_{-2}\frac{\hat{\Phi}_2}{\hat{\Phi}_1}\right].$$

The proposition follows the proof of Proposition 5 from here.

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