Computational Appendix of "Safe Assets"

Abstract

This appendix presents the equilibrium conditions of the model in the main part of the paper, explains the solution method, and references the `MATLAB` codes that replicate the results.

1 The Model

The economy consists of one Lucas tree and two types of agents, denoted $i \in \{1, 2\}$. The total size of the population is 2. The population share of type 1 agents is denoted N_1 and the population share of type 2 agents is, therefore, $1 - N_1$.

The utility of agent i, denoted U_{it}, is given by:

$$U_{it}^{1-\theta} = \frac{\rho}{1+\rho} C_{it}^{1-\theta} + \frac{1}{1+\rho} \mathbb{E}_t \left(U_{i,t+1}^{1-\gamma_i} \right)^{\frac{1-\theta}{1-\gamma_i}}. \tag{1}$$

Agent i's budget constraint is:

$$C_{it} + P_t K_{it} + B_{it} = (Y_t + P_t) K_{it-1} + R_t^f B_{it-1}, \tag{2}$$

where C_{it}, K_{it}, and B_{it} denote consumption, equity shares, and bonds determined in period t, P_t is the stock price and R_{t+1}^f is the risk-free interest rate paid in $t + 1$.

In equilibrium, aggregate consumption equals aggregate output, the aggregate stock of equity is 1, and the aggregate stock of bonds is 0:

$$2N_1 C_{1t} + 2(1 - N_1) C_{2t} = Y_t \tag{3}$$

$$2N_1 K_{1t} + 2(1 - N_1) K_{2t} = 1 \tag{4}$$

$$2N_1 B_{1t} + 2(1 - N_1) B_{2t} = 0. \tag{5}$$

Aggregate output follows the exogenous process:

$$\log Y_{t+1} = \log Y_t + g - bd_{t+1},$$

where d_{t+1} is a stochastic disaster shock with distribution:

$$d_t = \left\{ \begin{array}{c|c} 0 & 1-p \\ 1 & p \end{array} \right. . \tag{6}$$

The disaster probability is given by p and is assumed to be fixed.

The Euler conditions of agent i are given by:

$$\mathbb{E}_t \left(M_{it+1} R^f_{t+1} \right) = 1$$
$$\mathbb{E}_t \left(M_{it+1} \frac{Y_{t+1} + P_{t+1}}{P_t} \right) = 1,$$

where M_{it+1} denotes the stochastic discount factor of agent i, defined by:

$$M_{it+1} = \frac{1}{1+\rho} \frac{C_{it+1}^{-\theta}}{C_{it}^{-\theta}} \frac{U_{it+1}^{\theta-\gamma_i}}{\left(\mathbb{E}_t U_{it+1}^{1-\gamma_i}\right)^{\frac{\theta-\gamma_i}{1-\gamma_i}}}. \tag{7}$$

1.1 Random type changes

At the end of each period, agents' types change randomly with probability ν. Agents who change type become type 1 with probability N_1 and type 2 with probability $1 - N_1$. Hence, the population shares remain constant over time. The possibility of a type change does not affect agents' optimal decisions.

The type changes impose a transfer of assets between the two types of agents (we ignore distributional effects within types). Specifically, the number of type 1 agents that become type 2 is $2N_1\nu(1 - N_1)$. This is also the number of type 2 agents that become type 1. Hence, the amount of equity shares that is transferred from type 1 to type to is $2N_1\nu(1 - N_1)(K_{1t} - K_{2t})$. Per capita, the equity shares of type 1 agents decrease by $\nu(1 - N_1)(K_{1t} - K_{2t})$. Substituting in this expression the aggregate condition (4), we find that the per capita equity shares of agent 1 decrease by $\nu(K_{1t} - 0.5)$. Similar calculations imply that the per capita bond holdings of agent 1 decrease by νB_{1t}.

Let K^*_{1t} and B^*_{1t} denote equity and bond holdings of type 1 agents after the type change. It follows that:

$$K_{1t}^{*} = K_{1t} - \nu \left(K_{1t} - 0.5 \right) \tag{8}$$

$$B_{1t}^{*} = B_{1t} - \nu B_{1t}. \tag{9}$$

Thus, K_{1t}^{*} and B_{1t}^{*} comprise the portfolio of agent 1 at the beginning of period $t+1$. In the next pages, we omit the * notation, with the understanding that beginning of period variables should be interpreted as *ex post* the type change and end of period variables should be interpreted as *ex ante* the change.

1.2 Detrending

Define the risk-free assets held by agent 1 at the beginning of the period by A_{1t}:

$$A_{1t} = R_t^f B_{1t-1}.$$

Define the price of the risk-free asset by:

$$q_t^f = \frac{1}{R_{t+1}^f}.$$

We normalize the nonstationary variables by output. This gives the following equilibrium conditions:

$$
\begin{aligned}
\log \frac{Y_{t+1}}{Y_t} &= g - bd_{t+1} \\
\frac{C_{1t}}{Y_t} + \frac{P_t}{Y_t} K_{1t} + q_t^f \frac{A_{1t+1}}{Y_t} &= \left(1 + \frac{P_t}{Y_t}\right) K_{1t-1} + \frac{A_{1t}}{Y_{t-1}} \cdot \frac{Y_{t-1}}{Y_t} \\
\frac{C_{2t}}{Y_t} + \frac{P_t}{Y_t} K_{2t} + q_t^f \frac{A_{2t+1}}{Y_t} &= \left(1 + \frac{P_t}{Y_t}\right) K_{2t-1} + \frac{A_{2t}}{Y_{t-1}} \cdot \frac{Y_{t-1}}{Y_t} \\
\mathbb{E}_t \left(M_{1t+1} R_{t+1}^f\right) &= 1 \\
\mathbb{E}_t \left(M_{2t+1} R_{t+1}^f\right) &= 1 \\
\mathbb{E}_t \left(M_{1t+1} \frac{\frac{P_{t+1}}{Y_{t+1}} + 1}{\frac{P_t}{Y_t}} \cdot \frac{Y_{t+1}}{Y_t}\right) &= 1 \\
\mathbb{E}_t \left(M_{2t+1} \frac{\frac{P_{t+1}}{Y_{t+1}} + 1}{\frac{P_t}{Y_t}} \cdot \frac{Y_{t+1}}{Y_t}\right) &= 1 \\
K_{2t} &= \frac{1}{2\left(1 - N_1\right)} - \frac{N_1}{1 - N_1} K_{1t} \\
\frac{A_{2t}}{Y_{t-1}} &= -\frac{N_1}{1 - N_1} \frac{A_{1t}}{Y_{t-1}}
\end{aligned}
$$

together with the Epstein-Zin block for each $i = 1, 2$:

$$
\theta = 1: \ \log\left(\frac{U_{it}}{Y_t}\right) = \frac{\rho}{1+\rho} \log\left(\frac{C_{it}}{Y_t}\right) + \frac{1/(1+\rho)}{1-\gamma_i} \log \mathbb{E}_t \left(\left(\frac{U_{it+1}}{Y_{t+1}}\right)^{1-\gamma_i} \cdot \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\gamma_i}\right) \tag{10}
$$

$$
\theta \neq 1: \ \left(\frac{U_{it}}{Y_t}\right)^{1-\theta} = \frac{\rho}{1+\rho} \left(\frac{C_{it}}{Y_t}\right)^{1-\theta} + \frac{1}{1+\rho} \left[\mathbb{E}_t \left(\left(\frac{U_{it+1}}{Y_{t+1}}\right)^{1-\gamma_i} \cdot \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\gamma_i}\right)\right]^{\frac{1-\theta}{1-\gamma_i}}
$$

$$
M_{it+1} = \frac{1}{1+\rho} \frac{\left(\frac{C_{it+1}}{Y_{t+1}}\right)^{-\theta}}{\left(\frac{C_{it}}{Y_t}\right)^{-\theta}} \cdot \left(\frac{Y_{t+1}}{Y_t}\right)^{-\theta} \frac{\left(\frac{U_{it+1}}{Y_{t+1}}\right)^{\theta-\gamma_i} \cdot \left(\frac{Y_{t+1}}{Y_t}\right)^{\theta-\gamma_i}}{\left[\mathbb{E}_t \left(\left(\frac{U_{it+1}}{Y_{t+1}}\right)^{1-\gamma_i} \cdot \left(\frac{Y_{t+1}}{Y_t}\right)^{1-\gamma_i}\right)\right]^{\frac{\theta-\gamma_i}{1-\gamma_i}}}. \tag{11}
$$

1.3 Change of variables

For notational simplicity, define: $\tilde{C}_{it} = \frac{C_{it}}{Y_t}$, $\tilde{P}_t = \frac{P_t}{Y_t}$, $\tilde{U}_{it} = \frac{U_{it}}{Y_t}$. In addition, denote: $\hat{Y}_t = \frac{Y_t}{Y_{t-1}}$ and $\hat{A}_{1t} = \frac{A_{1t}}{Y_{t-1}}$. Last, the utility variables are scaled down and the time discount rate is given by $\beta = \frac{1}{1+\rho}$. This yields the following system:

$$\log \hat{Y}_{t+1} = g - bd_{t+1} \tag{12}$$

$$\tilde{C}_{1t} + \tilde{P}_t K_{1t} + q_t^f \hat{A}_{1t+1} = \left(1 + \tilde{P}_t\right) K_{1t-1} + \hat{A}_{1t}/\hat{Y}_t \tag{13}$$

$$\tilde{C}_{2t} + \tilde{P}_t K_{2t} + q_t^f \hat{A}_{2t+1} = \left(1 + \tilde{P}_t\right) K_{2t-1} + \hat{A}_{2t}/\hat{Y}_t \tag{14}$$

$$\mathbb{E}_t \left(M_{1t+1} R_{t+1}^f\right) = 1 \tag{15}$$

$$\mathbb{E}_t \left(M_{2t+1} R_{t+1}^f\right) = 1 \tag{16}$$

$$\mathbb{E}_t \left(M_{1t+1} \frac{\tilde{P}_{t+1} + 1}{\tilde{P}_t} \cdot \hat{Y}_{t+1}\right) = 1 \tag{17}$$

$$\mathbb{E}_t \left(M_{2t+1} \frac{\tilde{P}_{t+1} + 1}{\tilde{P}_t} \cdot \hat{Y}_{t+1}\right) = 1 \tag{18}$$

$$K_{2t} = \frac{1}{2\left(1 - N_1\right)} - \frac{N_1}{1 - N_1} K_{1t} \tag{19}$$

$$\hat{A}_{2t} = -\frac{N_1}{1 - N_1} \hat{A}_{1t}, \tag{20}$$

and the Epstein-Zin/Weil block that holds for each agent $i = 1, 2$:

$$F_{it}^{1-\gamma_i} = \mathbb{E}_t \left(\left(\tilde{U}_{it+1}/\tilde{U}_i\right)^{1-\gamma_i} \cdot \left(\hat{Y}_{t+1}\right)^{1-\gamma_i}\right) \tag{21}$$

$$\theta = 1: \ \log\left(\tilde{U}_{it}/\tilde{U}_i\right) = (1-\beta)\log\left(\tilde{C}_{it}/\tilde{C}_i\right) + (1-\beta)\log\left(\frac{\tilde{C}_i}{\tilde{U}_i}\right) + \beta \log F_{it} \tag{22}$$

$$\theta \neq 1: \ \left(\tilde{U}_{it}/\tilde{U}_i\right)^{1-\theta} = (1-\beta)\left(\tilde{C}_{it}/\tilde{C}_i\right)^{1-\theta} \left(\frac{\tilde{C}_i}{\tilde{U}_i}\right)^{1-\theta} + \beta F_{it}^{1-\theta}$$

$$M_{it+1} = \beta \frac{\left(\tilde{C}_{it+1}\right)^{-\theta}}{\left(\tilde{C}_{it}\right)^{-\theta}} \cdot \frac{\left(\tilde{U}_{it+1}/\tilde{U}_i\right)^{\theta-\gamma_i} \cdot \left(\hat{Y}_{t+1}\right)^{-\gamma_i}}{F_{it}^{\theta-\gamma_i}}. \tag{23}$$

$\tilde{U}_i$ and $\tilde{C}_i$ are scaling parameters for agent i. They are calibrated by the non-stochastic steady state values of the respective variables.

2 Computational Details

We cast the model into the following form:

$$\mathbb{E}_t f\left(y_{t+1}, y_t, x_{t+1}, x_t\right) = 0,$$
$$x_{t+1} = h\left(x_t\right) + \eta\epsilon_{t+1},$$
$$y_t = g\left(x_t\right)$$
$$h\left(x_t\right) = \begin{pmatrix} \tilde{h}\left(x_t\right) \\ \Phi\left(x_t^2\right) \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ \tilde{\eta} \end{pmatrix}.$$

y_t is a vector of control variables and $x_t = \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix}$ is a vector of state variables, where x_t^1 denotes the predetermined endogenous state variables and x_t^2 denotes the exogenous variables. η is a known matrix and ϵ_{t+1} is a vector of zero-mean shocks. The first rows of η that correspond to the predetermined state variables must be zero. The function Φ denotes the expected value of x_{t+1}^2. This function is known, because we have the law of motion of the exogenous state variables. The unknown functions are g and $\tilde{h}$ which provide the decision rules of y_t and x_{t+1}^1.

In our model, we define the model variables as:

$$x_t = \begin{pmatrix} K_{1t-1} \\ \hat{A}_{1t} \\ d_t \end{pmatrix}, \quad y_t = \begin{pmatrix} F_{1t} \\ F_{2t} \\ \log \tilde{C}_{1t} \\ \log \tilde{C}_{2t} \\ \log q_t^f \\ \log \tilde{P}_t \end{pmatrix}, \quad \Phi = p, \ \ \eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \epsilon_{t+1} = \left\{ \begin{array}{c|c} -p & 1-p \\ 1-p & p \end{array} \right. .$$

The state vector x_t contains two endogenous state variables and one exogenous variable (d_t).[1] The total number of endogenous variables is 8 (two state variables and 6 control variables). The other model variables can be expressed in terms of these variables through the model conditions.

2.1 Solution algorithm

The model is solved by the "Taylor projection" method described in Levintal (2016) and Fernández-Villaverde and Levintal (2016). Specifically, we approximate the policy functions g

[1]Note that the binary shock ϵ_{t+1} is demeaned. This will be useful to obtain a perturbation solution that will be used as an initial guess.

and $\tilde{h}$ by complete 4th-order polynomials and plug them into the model conditions to get the residual function $R(x, \Theta)$, where Θ denotes the vector of unknown polynomial coefficients. We evaluate R at the point of interest x_0 and use the Newton method to find Θ that zeros R and all its derivatives up to order k at x_0. Consequently, the 4th-order Taylor series of R about x_0 is zero, which implies that R is approximately zero in the neighborhood of x_0. Hence, Θ is an approximate solution to the model in this neighborhood. The approximate solution can be used as an initial guess for solving the model at a new point $x_1 \neq x_0$. Proceeding this way, we solve the model at different points of interest.

2.2 Initial guess

Finding Θ requires solving a nonlinear problem. We use the Newton method, for which we need a good initial guess. Usually, a perturbation solution is a good initial guess in the neighbourhood of the non-stochastic steady state. Hence, it is recommended to solve the model first at the non-stochastic steady state using the perturbation solution as an initial guess. Once the model is solved at the non-stochastic steady state, the solution can be used as an initial guess for solving the model at a different point in the state space.

The challenge that we face, however, is that the non-stochastic model (i.e., with zero volatility) does not have a stable perturbation solution because the wealth share of the two agents is indeterminate. To face this difficulty, we introduce a small cost parameter $\mu > 0$ into the optimization problem of agent 1, which pins down the non-stochastic steady state and yields a stable perturbation solution. We use this solution as an initial guess for the true model with $\mu = 0$.

Specifically, we change the budget constraint (2) of agent 1 as follows:

$$C_{it} + P_t K_{it} + B_{it} + .5\mu P_t \left(K_{1t} - \bar{K}_1\right)^2 + .5\mu q_t^f Y_t \left(\frac{B_{1t}}{q_t^f Y_t} - \bar{\hat{A}}_1\right)^2 = \left(Y_t + P_t\right) K_{it-1} + R_t^f B_{it-1},$$

where $\bar{K}_1$ and $\bar{\hat{A}}_1$ are parameters. Consequently, the budget constraint and the Euler conditions of agent 1, given by equations (13), (15) and (17), change into:

$$\tilde{C}_{1t} + \tilde{P}_t K_{1t} + q_t^f \hat{A}_{1t+1} + .5\mu \tilde{P}_t \left(K_{1t} - \bar{K}_1\right)^2 + .5\mu q_t^f \left(\hat{A}_{1,t+1} - \bar{\hat{A}}_1\right)^2 = \left(1 + \tilde{P}_t\right) K_{1t-1} + \hat{A}_{1t}/\hat{Y}_t$$
$$\mathbb{E}_t \left(M_{1t+1} R_{t+1}^f\right) = 1 + \mu \left(\hat{A}_{1,t+1} - \bar{\hat{A}}_1\right)$$
$$\mathbb{E}_t \left(M_{1t+1} \frac{\tilde{P}_{t+1} + 1}{\tilde{P}_t} \cdot \hat{Y}_{t+1}\right) = 1 + \mu \left(K_{1t} - \bar{K}_1\right).$$

Under this specification, the non-stochastic steady state equity and bond shares of agent 1 are

$\bar{K}_1$ and $\bar{\bar{A}}_1$, respectively. By choosing $\bar{K}_1 = 0.5$ and $\bar{\bar{A}}_1 = 0$, the non-stochastic steady state equity shares are equal across the two agents and bond issues are zero. The perturbation solution is stable, and we can employ it as an initial guess for solving the nonlinear problem at the initial point of the simulation, where we set $\mu = 0$ to remove the investment costs.

2.3 Simulations

To simulate the model we follow these steps:

1. We solve the model at the non-stochastic steady state (using a perturbation solution as an initial guess).

2. Starting at the non-stochastic steady state, the model is simulated along a path of no realized disasters. We check the accuracy of the solution along the simulation (measured by the model residuals) and solve the model again whenever accuracy falls below a threshold.

3. When the model reaches a fixed point, denoted x_0, the simulation is stopped. The fixed point x_0 is where the model converges in the absence of realized disasters. Hence, a simulation with disasters is likely to be in the neighborhood of x_0.

4. In addition, we solve the model at other points that the simulation is likely to visit. For instance, the state after a disaster hits (starting at x_0) is a point of high likelihood. Similarly, the state after a second disaster hits could also be likely.

5. Having solved the model at several points, we interpolate the solution of the model at a given point x_t using the available solutions. A simple Shepard's interpolation method performs well in this role. Specifically, suppose we want to approximate the value of y_t at x_t, given by the function $g(x_t)$. The model has already been solved at N different points $x_0, \ldots, x_{N-1}$, which implies that we have N different polynomials that approximate $g(x_t)$. We take a weighted average of these N approximations, where the weights are the inverse of the distance of x_t from the solution points $x_0, \ldots, x_{N-1}$.

6. Finally, we simulate the model with realized disasters, starting at x_0, for 2,000 years and compute the simulation moments reported in the main text. We monitor the accuracy of the solution by computing mean and max model residuals across the simulation. The accuracy measures are reported in Tables 1-3 below.

Table 1: Accuracy measures for Table 1 in the paper

γ_1	mean errors	max errors
	log10	
1	-15.5	-14.9
1.5	-10.7	-10.6
2	-9.1	-8.9
2.5	-8.0	-7.9
3	-12.9	-12.7
3.5	-11.7	-11.5
4	-10.6	-10.4
4.5	-9.6	-9.5
5	-8.7	-8.6
5.5	-7.9	-7.7
6	-7.0	-6.9

The table presents accuracy measures for Table 1 in the main text. The accuracy measures are the mean and maximum model residuals (presented in log10 units) across the simulations.

Table 2: Accuracy measures for Table 2 in the paper

γ_1	mean errors	max errors
	log10	
3.86	-10.9	-10.7
3.6	-5.7	-3.2
3.4	-5.1	-2.6
3.3	-4.4	-2.3
3.2	-3.3	-1.3
3.1	-2.6	-0.6

The table presents accuracy measures for Table 2 in the main text. The accuracy measures are the mean and maximum model residuals (presented in log10 units) across the simulations.

Table 3: Accuracy measures for Table 3 in the paper

	$\theta = .5$		$\theta = 2$		$\nu = .05$		$p = .02$		$N_1 = .25$	
γ_1	mean	max	mean	max	mean	max	mean	max	mean	max
					log10					
3.6	-5.6	-3.2	-5.4	-3.3	-5.3	-3.3	-5.8	-3.3	-5.6	-3.1
3.4	-5.0	-2.8	-4.5	-2.7	-4.6	-2.9	-5.4	-2.9	-4.3	-2.6
3.2	-4.1	-2.2	-1.3	0.0	-2.9	-1.2	-4.4	-2.2	-2.9	-0.9

The table presents accuracy measures for Table 3 in the main text. The accuracy measures are the mean and maximum model residuals (presented in log10 units) across the simulations.

3 Matlab codes

The companion Matlab package, available at http://economics.sas.upenn.edu/~jesusfv/Matlab solves the model by Taylor projection and replicates the results of the paper. Detailed installation instructions are described in the readme file. For a more general description of the Taylor projection algorithm and a simple example read the enclosed user guide.

References

Fernández-Villaverde, J., and O. Levintal (2016): "Solution Methods for Models with Rare Disasters," *Manuscript, University of Pennsylvania.*

Levintal, O. (2016): "Taylor Projection: A New Solution Method for Dynamic General Equilibrium Models," *Manuscript, Interdisciplinary Center Herzliya.*