Appendix C: Supplementary Proofs

Details for the proof of Theorem 5.1:

To see that the expectation of $\psi(Y,W,X)\cdot\mathcal{S}(Y,W,X|\theta_0)=\frac{\partial\tau_{P,\lambda}(\theta_0)}{\partial\theta}$ we write out all sixteen components of this product ($\psi(Y,W,X)$ has four terms, and $\mathcal{S}(Y,W,X|\theta_0)$ has four components), and then consider the expectations of all sixteen terms separately. First,

$$\psi(Y,W,X)\cdot s(Y,W,X|\theta_0)$$

$$= \frac{W^2\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_1(Y,X)(Y-\mathbb{E}[Y(1)|X]) \tag{C.1}$$

$$+\ 0 \tag{C.2}$$

$$+\ \frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_x(X)(Y-\mathbb{E}[Y(1)|X]) \tag{C.3}$$

$$+\ \frac{W(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)^2(1-e(X))}e'(X)(Y-\mathbb{E}[Y(1)|X]) \tag{C.4}$$

$$-\ 0 \tag{C.5}$$

$$-\ \frac{(1-W)^2\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_0(Y,X)(Y-\mathbb{E}[Y(0)|X]) \tag{C.6}$$

$$-\ \frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_x(X)(Y-\mathbb{E}[Y(0)|X]) \tag{C.7}$$

$$-\ \frac{(1-W)\cdot(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))^2}e'(X)(Y-\mathbb{E}[Y(0)|X]) \tag{C.8}$$

$$+\ \frac{W\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_1(Y,X)(\tau(X)-\tau_{P,\lambda}) \tag{C.9}$$

$$+\ \frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_0(Y,X)(\tau(X)-\tau_{P,\lambda}) \tag{C.10}$$

$$+\ \frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)(\tau(X)-\tau_{P,\lambda}) \tag{C.11}$$

$$+\ \frac{(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda}) \tag{C.12}$$

$$+\ \frac{W(W-e(X))\cdot\lambda'(e(x))}{\mu_\lambda}\mathcal{S}_1(Y,X)(\tau(X)-\tau_{P,\lambda}) \tag{C.13}$$

$$+\ \frac{(1-W)(W-e(X))\cdot\lambda'(e(x))}{\mu_\lambda}\mathcal{S}_0(Y,X)(\tau(X)-\tau_{P,\lambda}) \tag{C.14}$$

$$+\ \frac{(W-e(X))\cdot\lambda'(e(x))}{\mu_\lambda}\mathcal{S}_x(X)(\tau(X)-\tau_{P,\lambda}) \tag{C.15}$$

$$+\ \frac{(W-e(X))^2\cdot\lambda'(e(x))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda}). \tag{C.16}$$

Consider each expectation in turn. Equation (C.1) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{W^2\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_1(Y,X)(Y-\mathbb{E}[Y(1)|X])\right] \\
&\quad= \mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_1(Y(1),X)Y(1)\right] - \mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_1(Y(1),X)\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathbb{E}[W\cdot\mathcal{S}_1(Y(1),X)Y(1)|X]\right] \\
&\qquad -\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathbb{E}[W\cdot\mathcal{S}_1(Y(1),X)|X]\cdot\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_1(Y(1),X)Y(1)|X]\right] - \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_1(Y(1),X)|X]\cdot\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_1(Y(1),X)Y(1)|X]\right] \\
&\quad= \frac{1}{\mu_\lambda}\iint \lambda(e(x))y\mathcal{S}_1(y,x)f_1(y|x)f(x)dydx.
\end{aligned}
$$

Equation (C.3) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_x(X)(Y-\mathbb{E}[Y(1)|X])\right] \\
&\quad= \mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_x(X)Y(1)\right] - \mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_x(X)\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)}\mathcal{S}_x(X)\mathbb{E}[WY(1)|X]\right] - \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(1)|X]\right] - \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(1)|X]\right] \\
&\quad= 0.
\end{aligned}
$$

Equation (C.4) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{W(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)^2(1-e(X))}e'(X)(Y-\mathbb{E}[Y(1)|X])\right] \\
&\quad= \mathbb{E}\left[\frac{W(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)^2(1-e(X))}e'(X)(Y(1)-\mathbb{E}[Y(1)|X])\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)^2(1-e(X))}e'(X)\mathbb{E}[W(W-e(X))Y(1)|X]\right] \\
&\qquad -\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)^2(1-e(X))}e'(X)\mathbb{E}[W(W-e(X))|X]\cdot\mathbb{E}[Y(1)|X]\right] \\
&\quad= \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)}e'(X)\mathbb{E}[Y(1)|X]\right] - \mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)}e'(X)\mathbb{E}[Y(1)|X]\right] \\
&\quad= 0.
\end{aligned}
$$

Equation (C.6) yields,

$$
\begin{aligned}
&-\mathbb{E}\left[\frac{(1-W)^2\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_0(Y,X)(Y-\mathbb{E}[Y(0)|X])\right]\\
&\quad=-\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_0(Y(0),X)Y(0)\right]\\
&\qquad+\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_0(Y(0),X)\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathbb{E}[(1-W)\mathcal{S}_0(Y(0),X)Y(0)|X]\right]\\
&\qquad+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathbb{E}[(1-W)\mathcal{S}_0(Y(0),X)|X]\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_0(Y(0),X)Y(0)|X]\right]+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_0(Y(0),X)|X]\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_0(Y(0),X)Y(0)|X]\right]\\
&\quad=-\frac{1}{\mu_\lambda}\iint\lambda(e(x))y\mathcal{S}_0(y,x)f_0(y|x)f(x)dydx.
\end{aligned}
$$

Equation (C.7) yields,

$$
\begin{aligned}
&-\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_x(X)(Y-\mathbb{E}[Y(0)|X])\right]\\
&\quad=-\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_x(X)Y(0)\right]+\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_x(X)\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}\mathcal{S}_x(X)\mathbb{E}[(1-W)Y(0)|X]\right]+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(0)|X]\right]+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)\mathbb{E}[Y(0)|X]\right]\\
&\quad=0
\end{aligned}
$$

Equation (C.8) yields,

$$
\begin{aligned}
&-\mathbb{E}\left[\frac{(1-W)\cdot(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))^2}e'(X)(Y-\mathbb{E}[Y(0)|X])\right]\\
&\quad=-\mathbb{E}\left[\frac{(1-W)\cdot(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))^2}e'(X)(Y(0)-\mathbb{E}[Y(0)|X])\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))^2}e'(X)\mathbb{E}[(1-W)(W-e(X))Y(0)|X]\right]\\
&\qquad+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))^2}e'(X)\mathbb{E}[(1-W)(W-e(X))|X]\mathbb{E}[Y(0)|X]\right]\\
&\quad=-\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot(1-e(X))}e'(X)\mathbb{E}[Y(0)|X]\right]+\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda\cdot(1-e(X)))}e'(X)\mathbb{E}[Y(0)|X]\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.9) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_1(Y,X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{W\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_1(Y(1),X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[W\mathcal{S}_1(Y(1),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{e(X)\cdot\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_1(Y(1),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.10) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_0(Y,X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{(1-W)\cdot\lambda(e(X))}{\mu_\lambda}\mathcal{S}_0(Y(0),X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathbb{E}[(1-W)\mathcal{S}_0(Y(0),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{(1-e(X))\cdot\lambda(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_0(Y(0),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.11) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\lambda(e(X))}{\mu_\lambda}\mathcal{S}_x(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\frac{1}{\mu_\lambda}\int\lambda(e(x))(\tau(x)-\tau_{P,\lambda})\mathcal{S}_x(x)f(x)dx
\end{aligned}
$$

Equation (C.12) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{(W-e(X))\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\mathbb{E}[(W-e(X))|X]\cdot\lambda(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.13) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{W(W-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_1(Y,X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{W(W-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_1(Y(1),X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\lambda'(e(X))}{\mu_\lambda}\mathbb{E}[W(W-e(X))\mathcal{S}_1(Y(1),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{e(X)(1-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_1(Y(1),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.14) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{(1-W)(W-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_0(Y,X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{(1-W)(W-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_0(Y(0),X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\lambda'(e(X))}{\mu_\lambda}\mathbb{E}[(1-W)(W-e(X))\mathcal{S}_0(Y(0),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{e(X)(1-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathbb{E}[\mathcal{S}_0(Y(0),X)|X](\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.15) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{(W-e(X))\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_x(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\mathbb{E}[(W-e(X))|X]\cdot\lambda'(e(X))}{\mu_\lambda}\mathcal{S}_x(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=0.
\end{aligned}
$$

Equation (C.16) yields,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{(W-e(X))^2\cdot\lambda'(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{(W^2+e(X)^2-2\cdot We(X))\cdot\lambda'(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\mathbb{E}[W+e(X)^2-2\cdot We(X)|X]\cdot\lambda'(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{e(X)(1-e(X))\cdot\lambda'(e(X))}{\mu_\lambda\cdot e(X)(1-e(X))}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\mathbb{E}\left[\frac{\lambda'(e(X))}{\mu_\lambda}e'(X)(\tau(X)-\tau_{P,\lambda})\right]\\
&\quad=\frac{1}{\mu_\lambda}\int\lambda'(e(x))e'(x)[\tau(x)-\tau_{P,\lambda}]f(x)dx
\end{aligned}
$$

□

Lemma C.1 (Asymptotic Normality)

$$
\sqrt{N}\cdot(\hat{\tau}_{\lambda^*}-\tau_{S,\lambda^*})\xrightarrow{d}\mathcal{N}\left(0,\frac{1}{(\mathbb{E}[\lambda^*(X)])^2}\cdot\mathbb{E}\left[\lambda^*(X)^2\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}\right)\right]\right).
$$

Proof: By Lemma B.7, independent sampling, and because the second moment of $\phi(Y,W,X)$ exists, it follows that

$$
\sqrt{N}\cdot(\hat{\theta}-\theta_S)\xrightarrow{d}\mathcal{N}\left(0,\mathbb{E}\left[\phi(Y,W,X)^2\right]\right).
$$

In addition, $\sum_{i=1}^N\lambda^*(X_i)/N\to\mathbb{E}[\lambda^*(X)]=\mathbb{E}\left[e(X)\cdot(1-e(X))\right]$, so that

$$
\sqrt{N}\cdot(\hat{\tau}_{\lambda^*}-\tau_{S,\lambda^*})=\sqrt{N}\cdot\frac{1}{\sum_{i=1}^N\lambda^*(X_i)/N}\cdot\left(\hat{\theta}-\theta_S\right)\xrightarrow{d}\mathcal{N}\left(0,\frac{1}{(\mathbb{E}\left[e(X)\cdot(1-e(X))\right])^2}\cdot\mathbb{E}\left[\phi(Y,W,X)^2\right]\right).
$$

Next,

$$
\begin{aligned}
&\mathbb{E}\left[\phi(Y,W,X)^2\right]\\
&=\mathbb{E}\left[\lambda^*(X_i)^2\cdot\left(\left(\frac{Y\cdot W}{e(X)}-\mu_1(X)\right)-\left(\frac{Y\cdot(1-W)}{1-e(X)}-\mu_0(X)\right)-\left(\frac{\mu_1(X)}{e(X)}+\frac{\mu_0(X)}{1-e(X)}\right)\cdot(W-e(X))\right)^2\right]\\
&\quad=\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{Y\cdot W}{e(X)}-\mu_1(X)\right)^2\right]\\
&\qquad-2\cdot\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{Y\cdot W}{e(X)}-\mu_1(X)\right)\cdot\left(\frac{Y\cdot(1-W)}{1-e(X)}-\mu_0(X)\right)\right]\\
&\qquad-2\cdot\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{Y\cdot W}{e(X)}-\mu_1(X)\right)\cdot\left(\frac{\mu_1(X)}{e(X)}+\frac{\mu_0(X)}{1-e(X)}\right)\cdot(W-e(X))\right]\\
&\qquad+\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{Y\cdot(1-W)}{1-e(X)}-\mu_0(X)\right)^2\right]\\
&\qquad+2\cdot\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{Y\cdot(1-W)}{1-e(X)}-\mu_0(X)\right)\cdot\left(\frac{\mu_1(X)}{e(X)}+\frac{\mu_0(X)}{1-e(X)}\right)\cdot(W-e(X))\right]\\
&\qquad+\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\left(\frac{\mu_1(X)}{e(X)}+\frac{\mu_0(X)}{1-e(X)}\right)\cdot(W-e(X))\right)^2\right]\\
&\quad=\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\mu_1^2(X)\cdot\frac{1-e(X)}{e(X)}\right)\right]\\
&\qquad+\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot 2\cdot\mu_0(X)\cdot\mu_1(X)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(2\cdot\mu_0(X)\cdot\mu_1(X)+2\cdot\mu_1^2(X)\cdot\frac{1-e(X)}{e(X)}\right)\right]\\
&+\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{\sigma_0^2(X)}{1-e(X)}+\mu_0^2(X)\cdot\frac{e(X)}{1-e(X)}\right)\right]\\
&-\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(2\cdot\mu_0^2(X)\cdot\frac{e(X)}{1-e(X)}+2\cdot\mu_0(X)\cdot\mu_1(X)\right)\right]\\
&+\mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(2\cdot\mu_0(X)\cdot\mu_1(X)+\mu_1^2(X)\cdot\frac{1-e(X)}{e(X)}+\mu_0^2(X)\cdot\frac{e(X)}{1-e(X)}\right)\right]\\
=&\ \mathbb{E}\left[(e(X)\cdot(1-e(X)))^2\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}\right)\right],
\end{aligned}
$$

and the result in the Lemma follows. □

Proof of Theorem: 6.2: This follows from the combination of Lemmas B.6 and C.1. □

Proof of Theorem: 6.3: Again we apply Theorem 4.1 in Imbens and Ridder (2006), with as before the vector $\tilde{Y}$ defined as

$$
\tilde{Y}=\begin{pmatrix} Y\cdot W \\ Y\cdot(1-W) \\ W \end{pmatrix},
$$

Also define the functions $\omega:\mathbb{X}\mapsto\mathbb{R}$, $g:\mathbb{X}\mapsto\mathbb{R}^3$, and $m:\mathbb{R}^3\mapsto\mathbb{R}^2$:

$$
\omega(x)=1,
$$

$$
g(x)=\mathbb{E}[\tilde{Y}|X=x]=\begin{pmatrix} g_1(x)\\ g_2(x)\\ g_3(x)\end{pmatrix}=\begin{pmatrix}\mathbb{E}[Y\cdot W|X=x]\\ \mathbb{E}[Y\cdot(1-W)|X=x]\\ \mathbb{E}[W|X=x]\end{pmatrix}=\begin{pmatrix}\mu_1(x)\cdot e(x)\\ \mu_0(x)\cdot(1-e(x))\\ e(x)\end{pmatrix},
$$

$$
m(z)=\begin{pmatrix} z_1\cdot(1-z_3)-z_2\cdot z_3\\ z_3\cdot(1-z_3)\end{pmatrix},
$$

so that

$$
\frac{\partial}{\partial z'}m(z)=\begin{pmatrix} 1-z_3 & -z_3 & -z_1-z_2\\ 0 & 0 & 1-2z_3\end{pmatrix}.
$$

Then

$$
\theta=\mathbb{E}[\omega(X)\cdot m(g(X))]=\begin{pmatrix}\mathbb{E}\left[e(X)\cdot(1-e(X))\cdot\left(\frac{Y\cdot W}{e(X)}-\frac{Y\cdot(1-W)}{1-e(X)}\right)\right]\\ \mathbb{E}\left[e(X)\cdot(1-e(X))\right]\end{pmatrix}
$$

$$
=\begin{pmatrix}\mathbb{E}\left[(1-e(X))\cdot Y\cdot W-e(X)\cdot Y\cdot(1-W)\right]\\ \mathbb{E}\left[e(X)\cdot(1-e(X))\right]\end{pmatrix},
$$

and

$$
\hat{\theta}=\frac{1}{N}\sum_{i=1}^N\omega(X_i)\cdot m(\hat{g}(X_i)),
$$

so that

$$
\tau_{\lambda^*}=\theta_1/\theta_2,
$$

and

$$
\hat{\tau}_{\lambda^*}=\hat{\theta}_1/\hat{\theta}_2.
$$

Assumptions 3.1-3.2 and 6.1-6.3 imply Assumptions 3.2, 3.3, 4.1, and 4.2 in Imbens and Ridder (2006). Then by Theorem 4.1 in Imbens and Ridder (2006) we have

$$
\sqrt{N}\cdot\left(\hat{\theta}-\theta\right)=\frac{1}{\sqrt{N}}\sum_{i=1}^N\omega(X_i)\cdot\frac{\partial m}{\partial g'}(g(X))\left(\tilde{Y}-g(X)\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^N\omega(X_i)\left(m(g(X_i))-\theta\right)+o_p(1).
$$

Substituting in for $\omega(\cdot)$, $g(\cdot)$, $\tilde{Y}$, and $m(\cdot)$, leads to

$$\sqrt{N}\cdot\left(\hat{\theta}-\theta\right)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\begin{array}{c}\phi_1(Y_i,W_i,X_i)\\ \phi_2(Y_i,W_i,X_i)\end{array}\right),$$

where

$$\begin{aligned}\phi_1(y,w,x)&=(1-e(x))(y\cdot w-\mu_1(x)\cdot e(x))-e(x)\cdot(y\cdot(1-w)-\mu_0(x)\cdot(1-e))\\ &\quad -(w-e(x))\cdot(e(x)\cdot\mu_1(x)+(1-e(x))\cdot\mu_0(x))\\ &\quad +(1-e(x))\cdot\mu_1(x)\cdot e(x)-e(x)\cdot\mu_0(x)\cdot(1-e(x))-\theta_1,\end{aligned}$$

and

$$\phi_2(y,w,x)=(1-2\cdot e(x))\cdot(w-e(x))+e(x)\cdot(1-e(x))-\theta_2.$$

Let

$$\mathbb{V}=\left(\begin{array}{cc}\mathbb{V}_{11} & \mathbb{V}_{12}\\ \mathbb{V}_{12} & \mathbb{V}_{22}\end{array}\right)=\left(\begin{array}{cc}\mathbb{E}\left[\phi_1(Y,W,X)^2\right] & \mathbb{E}\left[\phi_1(Y,W,X)\cdot\phi_2(Y,W,X)\right]\\ \mathbb{E}\left[\phi_1(Y,W,X)\cdot\phi_2(Y,W,X)\right] & \mathbb{E}\left[\phi_2(Y,W,X)^2\right]\end{array}\right).$$

By a standard central limit theorem,

$$\sqrt{N}\cdot\left(\hat{\theta}-\theta\right)\xrightarrow{d}\mathcal{N}\left(0,\mathbb{V}\right),$$

and by the Delta method,

$$\sqrt{N}\cdot\left(\hat{\tau}_{\lambda^*}-\tau_{\lambda^*}\right)\xrightarrow{d}\mathcal{N}\left(0,\Omega\right),$$

where

$$\begin{aligned}\Omega&=\mathbb{V}_{11}/\theta_2^2-2\mathbb{V}_{12}\theta_1/\theta_2^3+\mathbb{V}_{22}\theta_1^2/\theta_2^4\\ &=\frac{\mathbb{E}\left[g(X)\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}+(\tau(X)-\tau_{P,g})^2\right)+e(X)(1-e(X))\cdot[h'(e(X))]^2(\tau(X)-\tau_{P,\lambda})^2\right]}{\mathbb{E}\left[g(X)\right]^2}.\end{aligned}$$

□

Proof of Theorem 6.4: First,

$$\hat{q}(\hat{\mathbb{A}})=\frac{N_{\hat{\mathbb{A}}}}{N}=\frac{N_{\mathbb{A}^*}}{N}+\frac{N_{\hat{\mathbb{A}}-N^*_{\hat{\mathbb{A}}}}}{N}.$$

The first term on the righthand side is equal to $\sum_{i=1}^{N}1\{X_i\in\mathbb{A}^*\}/N$, which converges to $q(\mathbb{A}^*)$ by a law of large numbers. The second term is $o_p(1)$ by Lemma B.4, so that the lefthand side converges to $q(\mathbb{A}^*)$.
Next, by the Triangle Inequality,

$$\left|\frac{1}{N_{\hat{\mathbb{A}}}}\sum_{i|X_i\in\hat{\mathbb{A}}}\left(\frac{\hat{\sigma}_1^2(X_i)}{\hat{e}(X_i)}+\frac{\hat{\sigma}_0^2(X_i)}{1-\hat{e}(X_i)}\right)-\mathbb{E}\left[\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}\right|X\in\mathbb{A}^*\right]\right|$$

$$\leq\left|\frac{1}{N_{\hat{\mathbb{A}}}}\sum_{i|X_i\in\hat{\mathbb{A}}}\left(\frac{\hat{\sigma}_1^2(X_i)}{\hat{e}(X_i)}+\frac{\hat{\sigma}_0^2(X_i)}{1-\hat{e}(X_i)}\right)-\frac{1}{N_{\hat{\mathbb{A}}}}\sum_{i|X_i\in\hat{\mathbb{A}}}\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}\right)\right| \tag{C.17}$$

$$+\left|\frac{1}{N_{\hat{\mathbb{A}}}}\sum_{i|X_i\in\hat{\mathbb{A}}}\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}\right)-\frac{1}{N_{\mathbb{A}^*}}\sum_{i|X_i\in\mathbb{A}^*}\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}\right)\right| \tag{C.18}$$

$$+\left|\frac{1}{N_{\mathbb{A}^*}}\sum_{i|X_i\in\mathbb{A}^*}\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}\right)-\mathbb{E}\left[\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}\right|X\in\mathbb{A}^*\right]\right| \tag{C.19}$$

All three terms are $o_p(1)$, so the result follows. □

Proof of Theorem: 6.5: By Assumption 3.2 $\inf_{x\in\mathbb{X}} \lambda^*(x) = \inf_{x\in\mathbb{X}} e(x)\cdot(1-e(x)) > 0$. By Assumptions 6.1-6.3 it follows that $\hat{\sigma}_w^2(x)$ and $\hat{e}(x)$ converge uniformly to their limits $\sigma_w^2(x)$ and $e(x)$. Hence

$$\hat{\mathbb{V}}_{S,\hat{\lambda}} = \frac{1}{\left(\frac{1}{N}\sum_{i=1}^N e(X_i)\cdot 1 - e(X_i)\right)^2}\cdot\frac{1}{N}\sum_{i=1}^N \left(e(X_i)\cdot(1-e(X_i))\right)^2\cdot\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}\right)+o_p(1)$$

$$= \frac{1}{\mathbb{E}\left[e(X)\cdot(1-e(X))\right]^2}\cdot\mathbb{E}\left[\left(e(X)\cdot(1-e(X))\right)^2\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}\right)\right]+o_p(1).$$

□

Proof of Theorem: 6.6: By Assumption 3.2 $\inf_{x\in\mathbb{X}} \lambda^*(x) = \inf_{x\in\mathbb{X}} e(x)\cdot(1-e(x)) > 0$. By Assumptions 6.1-6.3 it follows that $\hat{\sigma}_w^2(x)$, $\hat{\tau}(x)$ and $\hat{e}(x)$ converge uniformly to their limits $\sigma_w^2(x)$, $\tau(x)$, and $e(x)$. In addition, by Theorem 6.3 it follows that $\hat{\tau}_{S,\hat{\lambda}}$ converges to τ_{P,λ^*}. Hence

$$\hat{\mathbb{V}}_{P,\hat{\lambda}}$$

$$= \frac{\frac{1}{N}\sum_{i=1}^N \lambda^*(X_i)\cdot\left(\frac{\sigma_1^2(X_i)}{e(X_i)}+\frac{\sigma_0^2(X_i)}{1-e(X_i)}+(\tau(X_i)-\tau_{P,\omega})^2\right)+e(X_i)(1-e(X_i))\cdot\left[\lambda^{*\prime}(X_i)\right]^2(\tau(X_i)-\tau_{P,\omega})^2\right]}{\left(\frac{1}{N}\sum_{i=1}^N \lambda^*(X_i)\right)^2}+o_p(1)$$

$$= \frac{\mathbb{E}\left[\lambda^*(X)\cdot\left(\frac{\sigma_1^2(X)}{e(X)}+\frac{\sigma_0^2(X)}{1-e(X)}+(\tau(X)-\tau_{P,g})^2\right)+e(X)(1-e(X))\cdot\left[\lambda^{*\prime}(X)\right]^2(\tau(X)-\tau_{P,\omega})^2\right]}{\mathbb{E}\left[\lambda(X)\right]^2}+o_p(1).$$

□