

Capital Controls: Growth versus Stability*

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Abstract

This paper provides a unified theoretical framework to analyze the macroeconomic consequences of capital account liberalizations and capital controls, like capital inflow taxes. It identifies two pecuniary externalities that lead to inefficient outcomes in terms of welfare and to financial instability. The first externality undermines the “terms of trade hedge” while the second leads to excessive liquidity mismatch and leverage. Short-term debt flows, i.e. “hot money”, stabilize the economy up to a certain level of global imbalance, but expose the system to sudden stops and financial instability.

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1 Introduction

For a long time it was the Washington consensus to promote free trade and full capital account liberalization. A world in which goods and capital can flow freely was considered as the guiding north star and any incremental liberalization towards this ideal was considered as a step in the right direction. Recently, the IMF took on a more nuanced view, see Ostry, Ghosh, Habermeier, Chamon, Qureshi, and Reinhardt (2010).¹ This more balanced view acknowledges that in a second best world, liberalizing only some markets might be harmful. Especially, the build-up of persistent capital flow imbalances in form of short-term debt, referred to as “hot money” increases the risk of financial instability. To avoid sudden reversal it might be desirable to “manage” capital flows.

We develop a formal framework that allows one to analyze capital account liberalization, capital controls and other partial restrictions. We identify inefficiencies and instabilities that can arise from free debt capital flows. To this end we develop a dynamic two country, two good stochastic growth model in continuous time. The two consumption goods and the single physical capital good can be freely traded. Like in the classic Ricardian trade model, each country has some (comparative) advantage in producing one good and hence should ideally specialize in producing that good. Our model shows that in a world with less than perfect risk sharing (which can be justified by information problems and moral hazard consideration) opening the capital account can boost economic growth in normal times but the resulting build-up of global debt imbalances can be excessive and threaten financial stability.

Not having access to the world debt market limits a country’s ability to build up its capital stock (say after an adverse shock) and to produce goods for which it has a comparative advantage. This limits economic growth in normal times. The lack of access to financing does however have a favorable side-effect. By limiting output, the country enjoys favorable terms of trade - the price of its output good is relatively high. The lower the elasticity of substitution between the two output goods the larger is

¹See also the policy recommendations in the report of the Committee on International Economic Policy and Reform (2012).

natural terms of trade hedge. As a country opens its current account, debt financing enables each firm in the country to borrow and purchase more physical capital to produce more. However, by doing so, firms in the country jointly erode the price of their output good. Firms do not internalize this pecuniary externality which leads to an overall welfare loss in the economy and an outcome that is not constrained efficient. Moreover, increased leverage exposes firms in this country to further risk. In other words, while debt financing allows firms in different countries to specialize in their output goods for longer, it also increases the debtor firms' exposure to additional adverse shocks. For this reason there is in addition to the pecuniary externality in output prices, a second pecuniary externality with regard to future input prices of physical capital. When firms take on more leverage they become more exposed to further bad shocks. After a severe shock or sequence of bad shocks indebted firms try to reduce their debt level, cutting back their production scale by fire-selling their physical capital to the firms in the other country. Partial irreversibility of physical capital due to adjustment costs limit firms' ability to revert capital goods into consumption goods. Each firm in the country fully takes into account that it might have to fire-sell physical capital should further adverse shocks occur. However, they ignore that their actions as a group exacerbates this drop in value of physical capital. In a complete markets setting these pecuniary externalities have only second order effects, but in an incomplete markets setting they result in a constrained inefficient outcome. That is, a social planner that is limited to distort firms actions within the same constrained environment can increase overall welfare. Moreover, a simple wealth redistribution can be Pareto improving. For example, the inefficiency that results when creditor country firms abandon specialization can be mitigated by an (unexpected) debt forgiveness policy.

The market outcome is not only inefficient from a welfare perspective, it also leads to high volatility and instability. Markets are only partially stabilizing and have some destabilizing features. Free capital flows in form of debt are stabilizing only for small shocks as they soften their impact, but the associated increased leverage of the firms is destabilizing when firms face a sequence of adverse shocks or large shocks. Initially short-term *hot money* flows into the debtor country seemingly offsetting the initial

shocks, but they also make the country vulnerable to additional shocks. The initial calm is treacherous. Firms continue investing in only partially reversible capital goods and might suddenly face an outflow of the hot money. This leads to a *sudden stop*. Overall, capital account liberalization enhances the expected growth of the economy especially if the economy experiences only reasonably small shocks, but also introduces risk of sudden stops. In a sense short-term debt acts as a palliative. As the global imbalances build, things seem calm, prices of output and capital seem relatively stable. With a bit of luck positive shocks follows and the strains on the global economy may never be noticed. However, if another bad shock follows conditions deteriorate quickly. Exchange rate swings can be wild depending on the denomination of debt. In sum, an open capital account that allows the flow of short-term debt boosts economic growth and welfare in normal times, but makes the global economy susceptible to financial crisis. However, the latter can be controlled by avoiding excessive credit flows that arise due to pecuniary externalities.

In times of crisis a sudden unanticipated introduction of capital controls that dilute creditors' debt claims can be Pareto improving. Each individual creditor would be reluctant to sign on to this scheme, even though as a group all creditors are better off. To understand why, note that creditors are also consumers in our setting. Consumers benefit if the good is produced by the firms in countries that have the comparative advantage producing it. Our analysis also provides an explanation for the so-called "Phoenix miracle" a phenomenon that refers to the stylized fact that countries that suffered from a sudden stop returned relatively quickly to their previous growth path. It also explains why countries like Malaysia, which imposed capital controls, fared relatively better during and after the South East Asia crisis in the late 1990s.

Related Literature. Relative to the existing literature our framework makes several contributions. The model can be seen as a two country two good version of Brunnermeier and Sannikov (2013, 2011), which build in turn on the seminal contributions of Bernanke, Gertler, and Gilchrist (1999) and Kiyotaki and Moore (1997). Our paper clearly identifies two important pecuniary externalities. The terms of trade externality only arises in a multiple goods setting. That pecuniary externalities lead to

constrained inefficient outcome was first discussed in the general equilibrium literature (Stiglitz (1982), Geanakoplos and Polemarchakis (1986)). Our analysis also highlights that partially completing the market can lead to inferior outcomes, a result that was shown in the GE literature by Hart (1975).

Pecuniary fire sale externalities are the subject of extensive study in finance and international economics. In most models inefficiencies arise because the price move tightens an exogenously imposed collateral constraint, see e.g. Caballero and Krishnamurthy (2004). In Aoki, Benigno, and Kiyotaki (2009) collateral debt limits are lower for international lending than domestic lending arrangements.² In Bianchi (2011) and Mendoza (2010) and this constraint binds only occasionally potentially leading to sudden stops. Jeanne and Korinek (2011) proposes a Pigouvian tax to correct for the externality. In contrast, we do not impose any exogenous debt constraint. In our setting a sudden decline in debt arises endogenously due to the incomplete markets setting, as firms are limited in issuing equity claims. Costinot, Lorenzoni, and Werning (2012) derive the optimal capital flow tax for a country that tries to manipulate the terms in trade in order to extract monopoly rents from the other countries.

Our framework is general enough that it can have quantitative implications after some calibration. In this sense our model is closer to canonical international RBC model with capital formation as in Backus, Kehoe, and Kydland (1994). However, unlike Backus, Kehoe, and Kydland (1994) we consider an incomplete market setting. Hence, we cannot use the canonical macroeconomic approach of solving the planners' problem and then decentralizing the (global) economy. Like Obstfeld and Rogoff (1995) New Keynesian framework, we focus on short-term international debt. However, since our methodology does not restrict us to the log-linearization technique, we can also study crisis events far away from the steady-state.

The terms of trade hedge is related to the seminal paper by Cole and Obstfeld (1991) and can be traced back to the debate between Keynes (1929) and Ohlin (1929). We gain additional insights by varying the elasticity of substitution across both goods. In addition, we show that capital irreversibility requires an even greater hedge than

²In Maggiori (2013) countries differ in their financial developments rather than in producing different goods.

provided by a Cobb Douglas elasticity of substitution. Like our paper, Heathcote and Perri (2013) also allows for endogenous capital formation. The focus of their analysis is to replicate empirical patterns identified in the international business cycles literature. Since all the debt financing is short-term, our analysis also speaks to *hot money* in international capital flow and the fear of losing control of monetary policy by the monetary authority.

Empirical evidence about the effects of capital account liberalizations are mixed. See e.g. Obstfeld and Taylor (2004) or Magud, Reinhart, and Rogoff (2011). There are several examples where capital account liberalizations spurred growth but also other where it led to subsequent crises. Prominent examples are the Scandinavian crisis in the early 1990s the South East Asia crisis in the late 1990s. The terms sudden stop and Phoenix miracle were coined and empirically documented in Calvo (1998) and Calvo (2006).

2 The Model

In this section we develop a simple baseline model of a global economy that is populated by agents who live in two different countries, A and B . Both types of agents have the same preferences and can own capital. They can also both produce the two consumption goods a and b . Like in the classical Ricardian trade model agents in country A have a comparative advantage in producing product a , while agents in country B are better at producing good b . There are no trade barriers for the two output goods a and b as well as for the input good, physical capital. We focus on frictions in the international finance markets. In particular we contrast a global economy in which capital account is closed with a world in which the current account is open for short-term debt instruments. We also derive the benchmark outcome for the case when all contingent claims can be traded.

Technology. Capital can be used to produce goods a or b , which can be combined to produce the aggregate good. The aggregate good can be consumed, or used for investment to produce new capital.

When quantities y^a and y^b of goods a and b are combined, they make a total quantity

$$y = \left[\frac{1}{2}(y^a)^{\frac{s-1}{s}} + \frac{1}{2}(y^b)^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}}, \quad (1)$$

of the aggregate good. For $s = \infty$ both goods are perfect substitutes, for $s = 0$ there is no substitutability á la Leontieff, while for $s = 1$ the substitutability corresponds to the one of a Cobb-Douglas utility function. The index/aggregate good serves as numeraire and its price is normalized to one.

Agents in country A are better at producing good a , while agents in country B are better at producing b . From k_t units of capital, an agent in country A can produce good a at rate ak_t and good b at rate of only $\underline{a}k_t$, where $a > \underline{a} \geq 0$. Symmetrically, an individual in country B can produce good b at rate ak_t and good a at rate only $\underline{a}k_t$.

We denote the aggregate amount of world capital at time $t \in [0, \infty)$ by K_t . Denote the fraction of world capital is used by agents in country A to produce good a by ψ_t^{Aa} , the fraction used by agents in country B to produce good b by ψ_t^{Bb} , etc., so that

$$\psi_t^{Aa} + \psi_t^{Ab} + \psi_t^{Ba} + \psi_t^{Bb} = 1.$$

Then the total world supply of goods a and b is given by

$$Y_t^a = (\psi_t^{Aa}a + \psi_t^{Ba}\underline{a})K_t \quad \text{and} \quad Y_t^b = (\psi_t^{Bb}a + \psi_t^{Ab}\underline{a})K_t, \quad (2)$$

respectively. Then the total supply of the aggregate good is, naturally,

$$Y_t = \left[\frac{1}{2}(Y_t^a)^{\frac{s-1}{s}} + \frac{1}{2}(Y_t^b)^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}}, \quad (3)$$

and the prices of goods a and b in terms of the numeraire/aggregate good are

$$P_t^a = \frac{1}{2} \left(\frac{Y_t}{Y_t^a} \right)^{1/s} \quad \text{and} \quad P_t^b = \frac{1}{2} \left(\frac{Y_t}{Y_t^b} \right)^{1/s}. \quad (4)$$

Capital is subject to shocks, which depend on the technology in which the capital is employed. Also, new capital can be built through internal investment by using the

aggregate good. Overall, capital employed to produce good $i = a, b$ evolves according to

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma^i dZ_t^i, \quad (5)$$

where ι_t is the investment rate of the output good index per unit of capital (i.e., $\iota_t k_t$ is the total investment rate). Function Φ , which satisfies $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$, represents a standard investment technology with adjustment costs. In the absence of investment, capital managed by experts depreciates at rate δ . The concavity of $\Phi(\iota)$ represents *technological illiquidity*, i.e., adjustment costs of converting output to new capital and vice versa.

The two independent Brownian shocks dZ_t^a , dZ_t^b are exogenous shocks to the production technologies. Capital devoted to producing output good a is shocked by dZ_t^a , while capital devoted to producing good b is shocked by dZ_t^b .

Preferences. All agents in the world have identical risk and intertemporal preferences represented by the expected utility function

$$E \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right],$$

where c_t is the individual consumption of the aggregate good.

The parameter γ is the constant relative risk aversion coefficient and the inverse of the constant elasticity of intertemporal substitution. For the case of $\gamma = 1$, the utility function is given by $\log c$, and this case has particular tractability. The preference discount rate is given by ρ .

Markets for Physical Capital and the Risk-Free Bond. Individual experts and households can trade physical capital in a fully liquid international market. We denote the equilibrium market price of capital per unit (in terms of the aggregate output good) by q_t and postulate that its law of motion is of the form

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^{qa} dZ_t^a + \sigma_t^{qb} dZ_t^b. \quad (6)$$

That is, capital k_t is worth $q_t k_t$.

Absent capital controls there is also an international market for the risk-free bond, which is in zero net supply. Agents can go long (lend) or short (borrow) in the risk-free asset. The return on the risk-free asset is denoted by dr_t^F . In equilibrium both q_t and dr_t^F are determined *endogenously*.

Returns from Holding Physical Capital. The returns from capital depends on the identity of the agent who holds it and the technology that it is employed for. The capital gains from capital are given by $d(q_t k_t)/(q_t k_t)$, where k_t and q_t evolve as (5) and (6). The dividend yield from capital is given by $(aP_t^i - \iota_t)/q_t$ when it is used productively to produce good $i = a, b$, and by $(\underline{a}P_t^i - \iota_t)/q_t$ when it is employed by the agent to produce the good, to which the agent does not have a comparative advantage. Therefore, when an agent of type A uses capital to produce good a , he earns the return of

$$dr_t^{Aa} = \left(\frac{aP_t^a - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma^a \sigma_t^{qa} \right) dt + (\sigma^a + \sigma_t^{qa}) dZ_t^a + \sigma_t^{qb} dZ_t^b,$$

where we used Ito's lemma to compute the capital gains portion of the return, $d(q_t k_t)/(q_t k_t)$. The Ito-term $\sigma^a \sigma_t^{qa}$ reflects the covariance between exogenous volatility of capital stock a and the endogenous q risk price exposure. When agent A uses capital to produce good b , he earns

$$dr_t^{Ab} = \left(\frac{\underline{a}P_t^b - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma^b \sigma_t^{qb} \right) dt + \sigma_t^{qa} dZ_t^a + (\sigma^b + \sigma_t^{qb}) dZ_t^b,$$

etc. The optimal investment rate, which maximizes returns, is always given by the first-order condition $\Phi'(\iota) = 1/q_t$.

Financial Frictions and Capital Structure. There are financial frictions in this economy. We assume that absent capital controls agents can borrow through risk-free debt to buy capital, but cannot share risk of the capital they employ by issuing equity or through other means. A constraint on expert equity issuance can be justified in many ways, e.g., through the existence of an agency problem between the experts and

households. There is an extensive literature in corporate finance that argues that firm insiders must have some “skin in the game” to align their incentives with those of the outside equity holders.³ Typically, agency models imply that the expert’s incentives and effort increase along with the equity stake. The incentives peak when the expert owns the entire equity stake and borrows from outside investors exclusively through risk-free debt. For tractability, we make the extreme assumption that agents cannot issue any outside equity.

Each agent chooses his consumption rate, as well as the allocation of wealth to capital used to produce each good and to the risk-free asset. When agent $I = A, B$ consumes at rate $\zeta_t^I > 0$ and chooses to portfolio weights $(x_t^a, x_t^b, 1 - x_t^a - x_t^b)$, his net worth n_t evolves according to

$$\frac{dn_t}{n_t} = x_t^a dr_t^{Ia} + x_t^b dr_t^{Ib} + (1 - x_t^a - x_t^b) dr_t^F - \zeta_t^I dt. \quad (7)$$

Equation (7) (together with the solvency constraint $n_t \geq 0$) can be thought of as the agent’s budget constraint. Portfolio weights x_t^a and x_t^b must be nonnegative for all agents.

Definition. *An equilibrium is a map from any initial allocation of wealth as well as histories of shocks $\{Z_s^a, Z_s^b, s \in [0, t]\}$ to the allocation of capital $(\psi_t^{Aa}, \psi_t^{Ab}, \psi_t^{Ba}, \psi_t^{Bb})$ and consumption goods (C_t^A, C_t^B) as well as prices q_t and dr_t^F such that*

1. *all agents solve their optimal consumption and portfolio choice problems, subject to the budget constraints and*
2. *all markets clear, i.e.*⁴

$$\psi_t^{Aa} + \psi_t^{Ab} + \psi_t^{Ba} + \psi_t^{Bb} = 1 \quad \text{and} \quad C_t^A + C_t^B = Y_t - \iota_t K_t. \quad (8)$$

We denote the net worth of agents in country A at time t by N_t and the net worth

³See Jensen and Meckling (1976), Bolton and Scharfstein (1990), and DeMarzo and Sannikov (2006).

⁴If the markets for capital and aggregate output clear, then the market for the risk-free asset clears automatically by the Walras’ Law.

share (wealth share) of agents of country A , by $\eta_t \equiv N_t/(q_t K_t)$. Then the portfolio weights of representative agent A are given by

$$\left(\frac{\psi_t^{Aa}}{\eta_t}, \frac{\psi_t^{Ab}}{\eta_t}, 1 - \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \right),$$

and consumption rate, by $\zeta_t^A = C_t^A/N_t$. Likewise, for agents B , these are given by

$$\left(\frac{\psi_t^{Ba}}{1 - \eta_t}, \frac{\psi_t^{Bb}}{1 - \eta_t}, 1 - \frac{\psi_t^{Ba} + \psi_t^{Bb}}{1 - \eta_t} \right) \quad \text{and} \quad \zeta_t^B = \frac{C_t^B}{q_t K_t - N_t}.$$

Asset-pricing equations. Here, we take a technical detour to summarize equations that price available assets from the agent's consumption processes. If the consumption of representative agent A follows

$$\frac{dC_t^A}{C_t^A} = \mu_t^A dt + \sigma_t^{Aa} dZ_t^a + \sigma_t^{Ab} dZ_t^b,$$

then marginal utility $C_t^{-\gamma}$ follows

$$\frac{(dC_t^A)^{-\gamma}}{(C_t^A)^{-\gamma}} = \underbrace{\left(-\gamma \mu_t^A + \frac{\gamma(\gamma+1)}{2} ((\sigma_t^{Aa})^2 + (\sigma_t^{Ab})^2) \right)}_{M_t^A \equiv} dt - \gamma \sigma_t^{Aa} dZ_t^a - \gamma \sigma_t^{Ab} dZ_t^b.$$

This process can be used to write down the pricing conditions for any asset held by agents A . Specifically, for capital used to produce output a ,

$$M_t^A - \rho + \frac{E[dr_t^{Aa}]}{dt} \underbrace{-\gamma \sigma_t^{Aa} (\sigma^a + \sigma_t^{qa}) - \gamma \sigma_t^{Ab} \sigma_t^{qb}}_{\text{Cov} [dr_t^{Aa}, d(C_t^A)^{-\gamma}/(C_t^A)^{-\gamma}]} = 0. \quad (9)$$

Likewise, for capital used to produce good b ,

$$M_t^A - \rho + \frac{E[dr_t^{Ab}]}{dt} \underbrace{-\gamma \sigma_t^{Aa} \sigma_t^{qa} - \gamma \sigma_t^{Ab} (\sigma^b + \sigma_t^{qb})}_{\text{Cov} [dr_t^{Ab}, d(C_t^A)^{-\gamma}/(C_t^A)^{-\gamma}]} \leq 0, \quad (10)$$

with equality if agents A devote a positive amount of capital to produce good b , i.e. $\psi_t^{Ab} > 0$. Finally, the asset-pricing condition for the risk-free asset is

$$-\gamma\mu_t^A + \frac{\gamma(\gamma+1)}{2}((\sigma_t^{Aa})^2 + (\sigma_t^{Ab})^2) - \rho + \frac{dr_t^F}{dt} = 0. \quad (11)$$

Similar equations also hold for agents B.

First-best Benchmark. In the economy without frictions and complete markets full specialization realizes. Agents A specialize in producing only output good a and agents B only produce output good b . With the efficient allocation of capital to the production of two goods, $\psi_t^{Aa} = \psi_t^{Bb} = 1/2$ and $\psi_t^{Ab} = \psi_t^{Ba} = 0$. The total aggregate good

$$Y_t = \frac{aK_t}{2}$$

is divided between the two groups of agents according to their fixed Pareto weights $(\lambda, 1 - \lambda)$. For simplicity, we assume a symmetric economy, in which $\sigma^a = \sigma^b = \sigma$.

With complete markets agents would fully share the risks dZ_t^a and dZ_t^b . The price of capital and the risk-free rate are given by the following proposition.

Proposition 1 *With complete markets, the market outcome leads to the first best allocation with full specialization, $\psi_t^{Aa} = \psi_t^{Bb} = 1/2$, and output and investment rate levels are the same across both countries. The risk-free rate and the price of capital in a complete-market economy are time-invariant and given by*

$$r^F = \rho + \gamma(\Phi(\iota) - \delta) - \frac{\gamma(\gamma+1)\sigma^2}{4} \quad \text{and} \quad q = \frac{a/2 - \iota}{r^F + \gamma\sigma^2/2 - \Phi(\iota) + \delta}. \quad (12)$$

Proof. See Appendix A. ■

The risk-free rate is determined by the time-preference rate, the growth rate of capital and a covariance risk term. The price of capital is given by the Gordon growth formula, where the denominator is given by the risk-free rate plus a risk premium term minus the growth rate of capital.

3 Closed Capital Account

Let us now consider the case in which the capital account is closed. Agents in the economy cannot tap in the international debt market and cannot borrow. Then asset-pricing conditions (9) and (10) hold, but (11) becomes irrelevant since there is no risk-free asset.

The following proposition characterizes a procedure to compute equilibrium under the assumptions of logarithmic utility.

Proposition 2 *Suppose that all agents have logarithmic utility ($\gamma = 1$). Then the state space is divided into three regions. In the middle region, $\eta \in [\eta^\psi, 1 - \eta^\psi]$, all agents engage only in their most productive technology (full specialization), i.e. $\psi_t^{Aa} = \eta_t$, $\psi_t^{Bb} = 1 - \eta_t$ and $\psi_t^{Ba} = \psi_t^{Ab} = 0$, and the price of capital q_t satisfies*

$$\left[\frac{1}{2}(a\eta)^{\frac{s-1}{s}} + \frac{1}{2}(a(1-\eta))^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}} - \iota(q(\eta)) = \rho q(\eta). \quad (13)$$

In the region $[0, \eta^\psi]$, in which agents of country B use capital to produce good a, the price of capital and the production of agents B are determined jointly by the equations

$$\frac{aP^b - \underline{a}P^a}{\sigma^2 q(\eta)} = \frac{2\psi^{Bb}}{1-\eta} - 1 \quad \text{and}, \quad (14)$$

$$\left[\frac{1}{2}(a\eta + \underline{a}(1-\eta-\psi^{Bb}))^{\frac{s-1}{s}} + \frac{1}{2}(a\psi^{Bb})^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}} - \iota(q(\eta)) = \rho q(\eta). \quad (15)$$

At point η^ψ , ψ^{Bb} reaches $1 - \eta_t$ and $q(\eta)$ reaches the level defined by (13). The law of motion of η_t over the entire range $[0, 1 - \eta^\psi]$ is given by

$$\frac{d\eta_t}{\eta_t} = \left(\frac{aP_t^a - \iota_t}{q_t} + \psi^{Bb}(2\psi^{Bb} - 1)\sigma^2 - \rho \right) dt + \psi^{Bb}\sigma dZ_t^a - \psi^{Bb}\sigma dZ_t^b. \quad (16)$$

The region $(1 - \eta^\psi, 1]$, where agents A produce good b, is determined symmetrically to the region $[0, \eta^\psi]$.

Proof. See Appendix B. ■

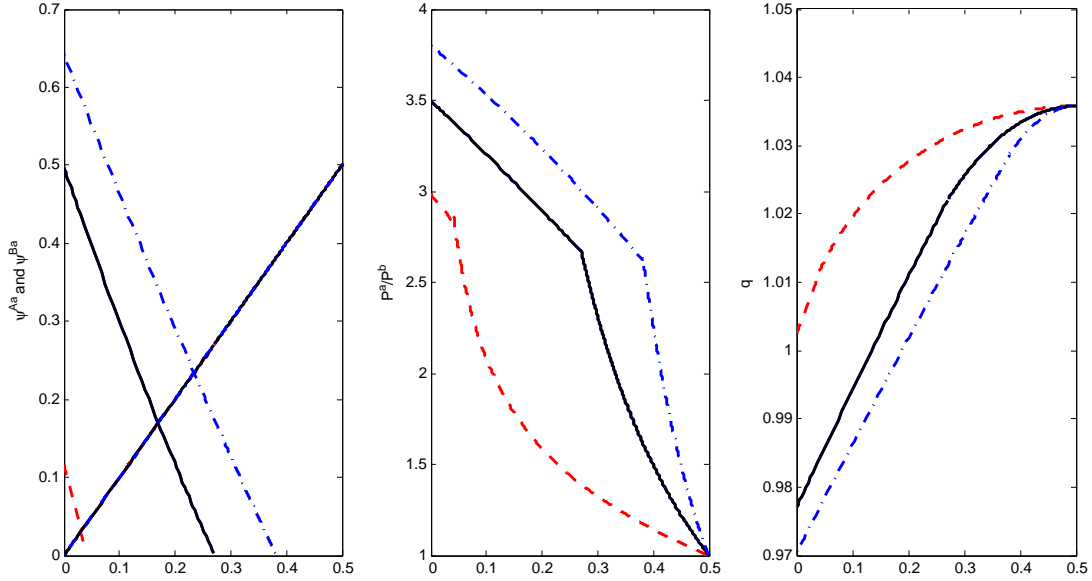


Figure 1: Panel A plots the **capital shares** ψ^{Aa} and ψ^{Ba} , Panel B plots the **terms of trade** P^a/P^b and Panel C plots the **price of physical capital** q for three different levels of elasticity of substitution: $s = 0.5$ in dash-dotted blue, $s = 1.01$ (Cobb-Douglas) in solid black, and $s = 3$ in dashed red.

Numerical Example. Our formal results allow us to characterize the full stochastic equilibrium dynamics of the global economy. For all numerical examples we assume that all agents have log utility function, i.e. $\gamma = 1$ and the adjustment cost function is given by $\Phi(i) = \frac{1}{\kappa} (\sqrt{1 + 2\kappa i} - 1)$ with $\kappa = 2$.⁵ We fix $\sigma^a = \sigma^b = 0.1$, $a = 0.14$, $\underline{a} = 0.04$, $\delta = 0.05$, $\kappa = 2$, $\rho = 0.05$ and vary the elasticity of substitution $s \in \{0.5, 1, 3\}$. Note that due to the symmetry of our setting it is sufficient to characterize the equilibrium for the wealth share $\eta \in [0, 0.5]$. For $\eta \in (0.5, 1]$ all functions of η are the mirror image.

Panel A of Figure 1 plots the capital share. Recall, the first best solution of complete markets resulted in *full specialization* with constant capital share of $\psi^{Aa} = 0.5$ and *full insurance* with a constant wealth share of $\eta = 0.5$. Capital controls modify this outcome. As long as $\eta \leq 0.5$ agents in country A still put all their wealth into producing output good a . However, as their wealth share declines, so does their capital share. That is, ψ^{Aa} is given by the 45-degree line, i.e. $\psi^{Aa} = \eta$ and their capital share equals their wealth share. This is the case independent of the elasticity of substitution s

⁵The investment technology in this example has quadratic adjustment costs: An investment of $\Phi + \frac{\kappa}{2}\Phi$ generates new capital at rate Φ .

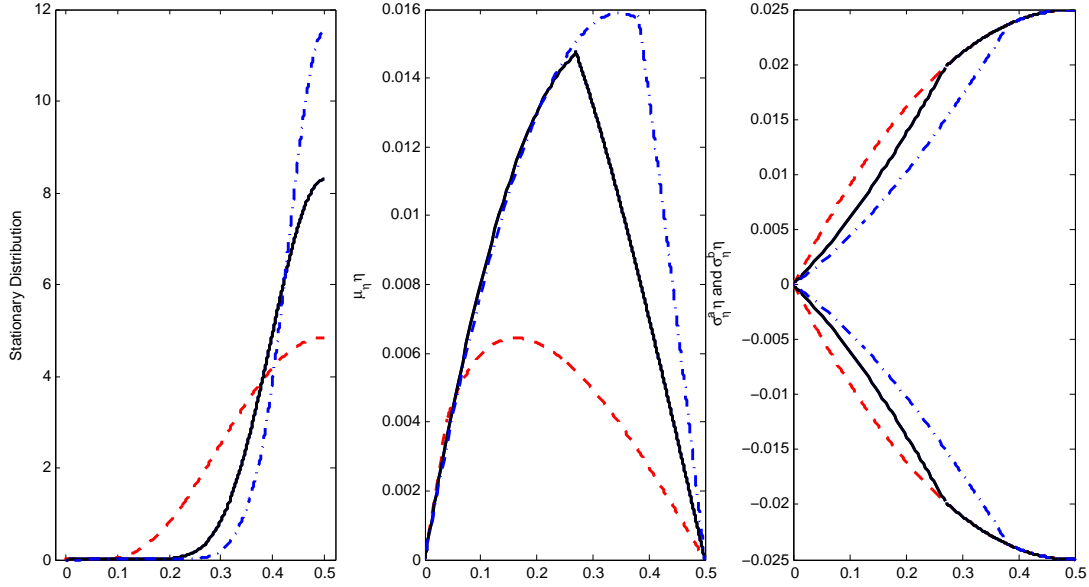


Figure 2: Panel A plots the **stationary distribution**, Panel B the **drift** and Panel C the **volatility** of **wealth shares** η for three different levels of elasticity of substitution: $s = 0.5$ in dash-dotted blue, $s = 1.01$ (Cobb-Douglas) in solid black, and $s = 3$ in dashed red.

between both output goods. The three decreasing lines ψ^{Ba} depict how much capital agent B devotes to producing output good a as a fraction of global capital K_t for different elasticities of substitutions s . The solid black line captures the case of Cobb Douglas aggregation, i.e. $s = 1$, the dash-dotted blue line the case of $s = 0.5$ and the dashed red line covers the case of $s = 3$. For η values sufficiently close to 0.5, $\psi^{Ba} = 0$. The declining lines cross the x-axis at the point η^ψ .

Panel B of Figure 1 plots the terms of trade, i.e. the price ratio P^a/P^b . The figure shows clearly the “*terms of trade hedge*.” As agent A ’s wealth share drops after a negative shock, its output good a becomes more scarce. Consequently, the price of this good rises and agent A ’s terms of trade improve. The improvement in the terms of trade increases as both output goods become worse substitutes, i.e. for decreasing s .

Panel C of Figure 1 shows the price of physical capital q for the three different cases of elasticity of substitutions. Recall that the reinvestment rate per unit of capital ι is directly related to the price of capital q , through the first order condition $\Phi'(\iota) = 1/q_t$. A higher capital price q translates in a higher investment rate.

Figure 2 fully characterizes the stochastic dynamics of the state variable, agent A ’s wealth share, η . Panel A plots the stationary distribution for $\eta \in [0, 0.5)$. Two features

stand out. First, lower output good substitutability leads to a tighter distribution of wealth shares. The power of the “terms of trade hedge” decreases with the elasticity of substitution s . Recall that the first best complete market solution implies a stationary distribution that is concentrated at $\eta = 0.5$. Second, unlike in Cole and Obstfeld (1991) the Cobb-Douglas case does not lead to full insurance nor to the first best outcome in our setting. What explains this difference? In our production economy an adverse shock destroys part of agents A ’s capital stock. Hence, the terms of trade have to hedge (i) agents A ’s loss in wealth devoted to consumption and (ii) agents A ’s loss to replace the capital stock. Cole and Obstfeld’s analysis does not include the latter aspect. Therefore, their conclusion that they have resolved the “international diversification puzzle” is not true for our production economy setting.

Macroeconomists typically represent the stochastic dynamics of an economy with impulse response functions. However, impulse response functions only plot the expected response of a variable from a specific starting point (e.g. the steady state). The drift and volatility of the state variable, depicted in Panel B and C of Figure 2 allows a full characterization of the dynamical system, independent of the starting point. The drift of η (Panel B) reveals that the system has a basin of attraction at $\eta = 0.5$. Whenever the system falls below $\eta = 0.5$, the positive drift pushes it back towards $\eta = 0.5$. Similarly, for $\eta > 0.5$, the system drifts back to $\eta = 0.5$ as the drift is negative in the range $\eta \in (0.5, 1)$. Panel C depicts the volatility σ_{η}^a and σ_{η}^b .

Figure 3 plots the frontier of value functions for agents in country A on the x-axis and for agents in country B on the y-axis. The outer frontier, given by the pink dotted curve, depicts the first best outcomes that arise as equilibrium outcome under complete markets. The values themselves turn out to be negative, which is not surprising given that the value functions are of the log form like the utility function. Recall, under first best the starting wealth share stays constant and determines Pareto weight of agents in country A and the point on the pink outer frontier. The frontier is also the Pareto frontier as one moves along the frontier agents A ’s value increases only when agents B ’s value declines. The other three “frontiers” cover the case of capital controls for different elasticity of substitutions. Surprisingly, a lower elasticity of substitutions shrinks the frontier payoffs for most η values, even though the “terms of trade hedge”

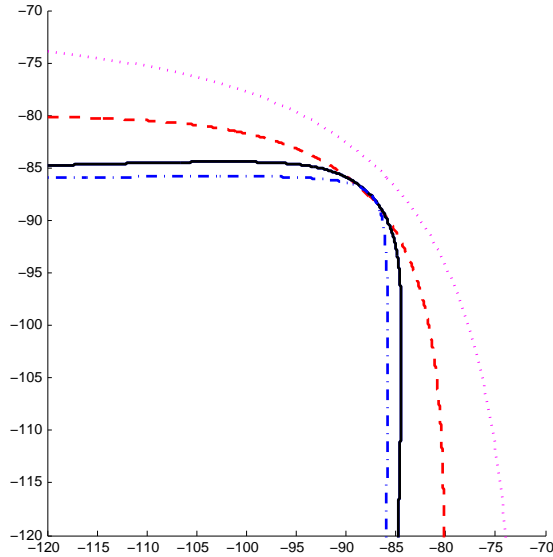


Figure 3: Welfare frontier for agents in country A (x-axis) and agents in country B (y-axis) for the first-best solution given by the dotted pink curve and for the economy with capital controls for three different levels of elasticity of substitution: $s = 0.5$ in dash-dotted blue, $s = 1.01$ (Cobb-Douglas) in solid black, and $s = 3$ in dashed red.

is more powerful. The figure reveals another interesting fact, the frontiers are inward bending for low enough s and hence are not necessarily Pareto frontiers, a phenomenon we will focus on in greater detail in the following section.

4 Open Capital Account for Debt

In this section, we discuss the equilibrium in an economy, in which agents in both countries can borrow through risk-free debt to buy capital. Due to incentive constraints they still cannot issue equity. Relative to the benchmark without the international debt market, agents do not have to cut back on production following negative shocks to their net worth. They can maintain their production level by borrowing.⁶ This has several effects. On one hand, efficiency is temporarily improved. On the other hand, the undercapitalized sector has greater risk exposure and benefits less from the terms of trade hedge (i.e. the price of its good rises to a lesser extent when it becomes undercapitalized). The combination of these factors increases the chance that the constrained sector becomes severely undercapitalized. We find that these extreme

⁶In equilibrium, agents will choose to cut down their production slowly after negative shocks, in order to manage risks.

regimes are Pareto inferior. That is, the overcapitalized sector can improve its welfare by forgiving some debt from the undercapitalized sector.

Technically, the equilibrium with debt is characterized by the asset pricing equations (9), (10) and (11) for agents of type A, together with analogous equations for agents of type B, the market-clearing conditions (8) and equations (2), (3) and (4) that determine output and prices. We use these equations, together with the law of motion of η_t given by Proposition 3, to solve for the equilibrium quantities as functions of the wealth allocation, summarized by η_t .

Proposition 3 *The law of motion of η_t is given by*

$$\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \gamma \frac{1 - \eta_t}{\eta_t} \left(\psi_t^{Aa} \sigma^a \sigma^{Aa} + \psi_t^{Ab} \sigma^b \sigma^{Ab} + (\psi_t^{Aa} + \psi_t^{Ab})(\sigma_t^{qa} \sigma_t^{Aa} + \sigma_t^{qb} \sigma_t^{Ab}) \right) dt \\
&- \gamma \left(\psi_t^{Ba} \sigma^a \sigma^{Ba} + \psi_t^{Bb} \sigma^b \sigma^{Bb} + (\psi_t^{Ba} + \psi_t^{Bb})(\sigma_t^{qa} \sigma_t^{Ba} + \sigma_t^{qb} \sigma_t^{Bb}) \right) dt + \frac{Y_t - \iota K_t}{q_t K_t} dt \\
&- \frac{C_t}{N_t} dt - \left(((\psi_t^{Aa} + \psi_t^{Ba}) \sigma^a + \sigma_t^{qa}) \sigma^{\eta a} + (\sigma_t^{qb} + (\psi_t^{Ab} + \psi_t^{Bb}) \sigma^b) \sigma^{\eta b} \right) dt \\
&+ \underbrace{\left(\left(\frac{\psi_t^{Aa}}{\eta_t} - \psi_t^{Aa} - \psi_t^{Ba} \right) \sigma^a + \frac{\psi_t^{Aa} + \psi_t^{Ab} - \eta_t}{\eta_t} \sigma_t^{qa} \right)}_{\sigma^{\eta a}} dZ_t^a \\
&+ \underbrace{\left(\left(\frac{\psi_t^{Ab}}{\eta_t} - \psi_t^{Ab} - \psi_t^{Bb} \right) \sigma^b + \frac{\psi_t^{Ab} + \psi_t^{Bb} - \eta_t}{\eta_t} \sigma_t^{qb} \right)}_{\sigma^{\eta b}} dZ_t^b. \tag{17}
\end{aligned}$$

Proof. See Appendix C. ■

We can use equation (17) to evaluate the volatilities of η and q from the capital allocation vector $(\psi_t^{Aa}, \psi_t^{Ab}, \psi_t^{Ba}, \psi_t^{Bb})$ and the values of $q(\eta)$ and $q'(\eta)$. Indeed, using Ito's lemma,

$$\sigma_t^{qa} = \frac{q'(\eta)}{q(\eta)} \underbrace{\left((\psi_t^{Aa} - \eta_t(\psi_t^{Aa} + \psi_t^{Ba})) \sigma^a + (\psi_t^{Aa} + \psi_t^{Ab} - \eta_t) \sigma_t^{qa} \right)}_{\sigma_t^{\eta a} \eta} \Rightarrow$$

$$\sigma_t^{\eta a} \eta = \frac{\psi_t^{Aa} - \eta_t(\psi_t^{Aa} + \psi_t^{Ba})}{1 - \frac{q'(\eta)}{q(\eta)}(\psi_t^{Aa} + \psi_t^{Ab} - \eta_t)} \sigma^a, \quad \sigma_t^{qa} = \frac{q'(\eta)}{q(\eta)} \sigma_t^{\eta a} \eta.$$

For the special case of logarithmic utility, we can also explicitly evaluate the drift of η . Indeed, since the consumption of all agents is proportionate to their net worth, we have

$$\begin{aligned} \sigma_t^{Aa} &= \frac{\psi_t^{Aa}}{\eta_t} \sigma^a + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qa}, & \sigma_t^{Ab} &= \frac{\psi_t^{Ab}}{\eta_t} \sigma^b + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qb} \\ \sigma_t^{Ba} &= \frac{\psi_t^{Ba}}{1 - \eta_t} \sigma^a + \frac{\psi_t^{Ba} + \psi_t^{Bb}}{1 - \eta_t} \sigma_t^{qa} & \text{and} & \quad \sigma_t^{Bb} = \frac{\psi_t^{Bb}}{1 - \eta_t} \sigma^b + \frac{\psi_t^{Ba} + \psi_t^{Bb}}{1 - \eta_t} \sigma_t^{qb} \end{aligned}$$

We can use these expressions in (17) directly.

We can combine the law of motion of η together with the equilibrium conditions (9), (10) and (11) for agents of type A, their equivalents for agents B, and the market-clearing conditions (8) to compute equilibrium dynamics and welfare. We outline the numerical procedure that we employ in Appendix For the case of logarithmic utility, the market-clearing condition for output takes the form

$$\rho q_t = \left[\frac{1}{2} (\psi_t^{Aa} a + \psi_t^{Ba} \underline{a})^{\frac{s-1}{s}} + \frac{1}{2} (\psi_t^{Bb} a + \psi_t^{Ab} \underline{a})^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}} - \iota(q_t). \quad (18)$$

Equilibrium dynamics and Welfare. As before we illustrate our findings within a specific numerical examples. To ease the comparison with the previous section we apply the same parameter values. Instead of focusing on different levels of substitutions, in this section we stress the difference between outcomes under an open and closed capital account. Recall that we do not allow risk sharing for incentive reasons.

Panel A of Figure 4 shows the difference between a global economy with and without capital controls. For equal wealth share, i.e. for $\eta = 0.5$, agents are in both cases fully specialized. However, as η , the wealth share for A , declines agents A in the economy with capital controls have to cut back their production. In contrast, with international debt market agents in country A continue to produce a large scale. They can do so since they can issue short-term debt to agents in country B . Panel A reveals that ψ^{Aa} is significantly higher in the case without capital controls (increasing red dashed curve) than in the case with capital controls (black increasing 45-degree line). In short,

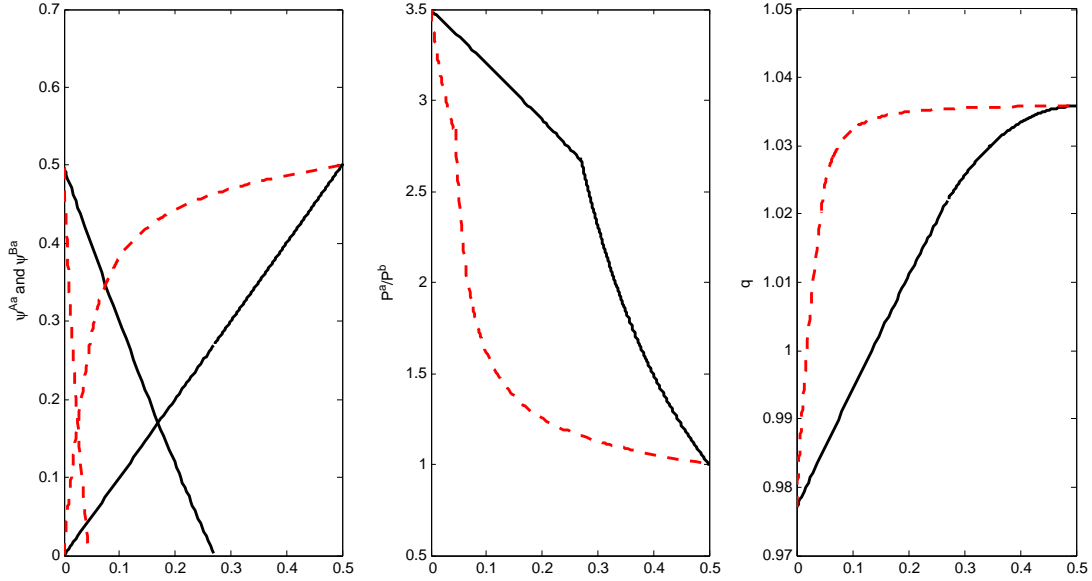


Figure 4: Panel A, B and C contrast the **capital shares** ψ^{Aa} and ψ^{Ba} , the **terms of trade** P^a/P^b and the **price of physical capital** q of an economy with and without capital controls (black solid versus red dashed curves) assuming an elasticity of substitution of $s = 1.01$ (close to Cobb-Douglas).

specialization can be maintained for a much larger range of η -values. Agents in country B start producing the output good a only for much smaller η values. The contrast between the two ψ^{Ba} s lines, the declining red dashed curve for the case open capital accounts and the black line for capital control economy is striking.

Panel B shows the difference in the terms of trade, the relative price ratio P^a/P^b . Without capital controls each agent in country A borrows in order to hold a larger fraction of the global physical capital stock, resulting in greater output of good a . This ruins the terms of trade improvement that would occur with capital controls. A pecuniary externality arises. Each individual agent ignores the effect of his production on his fellow countrymen; by borrowing and operating at a larger scale, output increases and the price appreciates less. Easier debt financing also pushes up the price of physical capital q as shown in Panel C of Figure 4.

Overall, debt financing increases specialization, it leads to better allocation of resources and boosts economic growth in normal times. However, it comes at the price of reduced economic stability. Panel A of Figure 5 shows this. The stationary distribution of the wealth share is more fat tailed without capital controls. Panel C explains why. The volatility with an open capital account is now much higher for low and mid-range

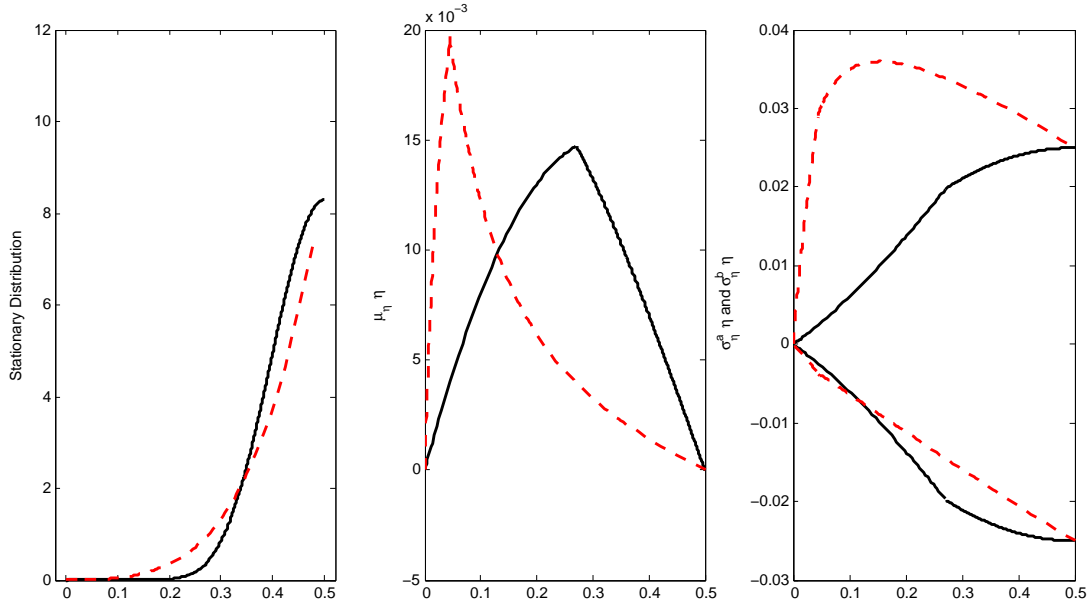


Figure 5: Panel A plots the **stationary distribution**, Panel B the **drift** and Panel C the **volatility** of **wealth shares** η for the global economy with capital controls (black solid curves) and without capital controls (red dashed curves) for an elasticity of substitution coefficient of $s = 1.01$.

η -values.

The examination of the value frontier is most interesting. As in Figure 3 the pink dotted outer frontier represent the first best outcomes. The inner frontier curve corresponds to the case of capital controls with $s = 1$ identical to Figure 3. The red dashed line in between is the frontier without capital controls. Interestingly, the frontier is inward bending. That is, for low enough values of η an unanticipated wealth transfer from agents in country B to agents in country A can make both types of agents better off. In other words, an unexpected *bailout* of or *debt relief* for agents A can be a Pareto improvement. (By symmetry, the reverse transfer can lead to Pareto improvement for high enough starting η .) This may be less surprising, after one takes into account that two forms of pecuniary externalities operate in the absence of capital controls. To test the importance of the “terms of trade externality” we lower the elasticity of substitution between both output goods to $s = 0.5$. In this case, the “terms of trade hedge” would be more pronounced, but is collectively undermined by agents in country A since debt financing allows them to keep their output ratio relatively high.

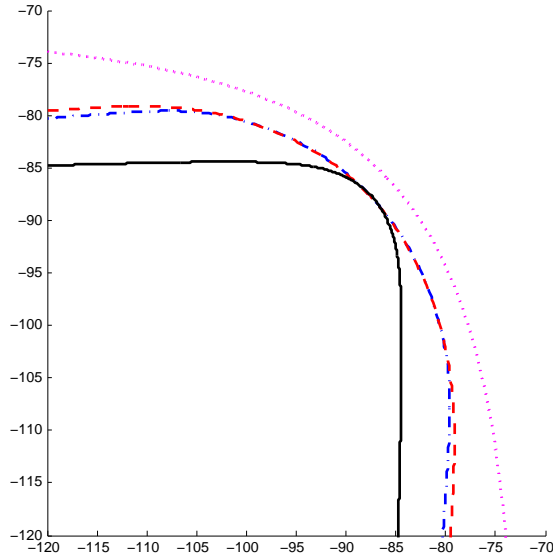


Figure 6: Welfare frontier for agents in country A (x-axis) and agents in country B (y-axis) for the first-best solution (pink dotted frontier) and for the economy with capital controls (inner black solid frontier). The middle curves depict the value frontier for a global economy without capital controls and an elasticity of substitution of $s = 1.01$ (red dashed curve) and of $s = 0.5$ (blue dashed curve).

5 Conclusion

Magud, Reinhart, and Rogoff (2011) complain about the lack of a unified theoretical framework to analyze the macroeconomic consequences of capital controls. This paper provides such a framework that is general enough that it can be calibrated and quantitative implications can be derived. It clearly identifies two pecuniary externalities. The first externality only arises in a multiple good setting and undermines the natural terms of trade hedge stressed in Cole and Obstfeld (1991). The second externality arises in a multiple period setting and is related to the fire-sale externality. Open current accounts that primarily lead to short-term debt financing also lead to a constrained inefficient outcome in terms of welfare and to a highly volatile market in terms of financial stability. Interestingly, in times of a crisis, unanticipated bail-out arrangements in favor of the debtor countries can be Pareto improving. That is the consumers in the creditor country also benefit from the bailout even if they have to pay for it. Our framework can form the basis to analyze numerous other policy measures.

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Appendix

A Proof of Proposition 1: First Best Analysis

Social Planner's Problem

Under complete market and no frictions, agents would fully share the risks dZ_t^A and dZ_t^B , and the equilibrium allocation solves a planner's problem, where welfare weights λ^A and $1 - \lambda^A$ depend on relative initial wealth of agents in country A and B :

$$\begin{aligned} \bar{V}(K_0) = & \max_{c_t, k_t^A, k_t^B, \iota_t, \psi_t, Y_t} E_0 \left[\int_0^\infty e^{-\rho t} [\lambda^A U(c_t^A) + (1 - \lambda^A) U(c_t^B)] dt \right] \\ & \text{subject to: } c_t^A + c_t^B + \iota_t^A k_t^A + \iota_t^B k_t^B = Y_t, \\ & k_t^A + k_t^B = K^t, k_t^A \geq 0, k_t^B \geq 0, \\ & y_t^{a,A} = a \psi_t^A k_t^A, \\ & y_t^{b,A} = \underline{a} (1 - \psi_t^A) k_t^A, \\ & y_t^{a,B} = \underline{a} (1 - \psi_t^B) k_t^B, \\ & y_t^{b,B} = a \psi_t^B k_t^B, \\ & Y_t = \left[\frac{1}{2} \left(y_t^{a,A} + y_t^{a,B} \right)^{\frac{s-1}{s}} + \frac{1}{2} \left(y_t^{b,A} + y_t^{b,B} \right)^{\frac{s-1}{s}} \right]^{\frac{s}{s-1}}, \\ & dK_t = [\Phi(\iota_t^A) - \delta] k_t^A dt + \sigma k_t^A \psi_t^A dZ_t^a + \sigma k_t^A (1 - \psi_t^A) dZ_t^b \\ & \quad + [\Phi(\iota_t^B) - \delta] k_t^B dt + \sigma k_t^B (1 - \psi_t^B) dZ_t^a + \sigma k_t^B \psi_t^B dZ_t^b \end{aligned} \tag{P1}$$

The social planner will choose **full specialization** ($\psi_t^A = \psi_t^B = 1$). Since marginal cost of goods a and b are identical (a times shadow rent of capital), marginal product of goods a and b in producing output-index Y must be the same. Write out the marginal products to see that the social planner must also choose **output equalization** ($y_t^a = y_t^b$) and **input equalization** ($k_t^A = k_t^B = \frac{K_t}{2}$); and for minimizing capital adjustment costs, **investment rate equalization** ($\iota_t^A = \iota_t^B$). The aggregate production function will be $Y_t = aK_t/2$. Let $c_t^A \equiv \zeta_t^A K_t$ and $c_t^B \equiv \zeta_t^B K_t$. The social planner's problem

above reduces to:

$$\begin{aligned} \bar{V}(K_0) &= \max_{\zeta_t^A, \zeta_t^B \geq 0} E_0 \left[\int_0^\infty e^{-\rho t} [\lambda^A U(\zeta_t^A K_t) + (1 - \lambda^A) U(\zeta_t^B K_t)] dt \right] \\ \text{subject to: } dK_t &= \left[\Phi\left(\frac{a}{2} - \zeta_t^A - \zeta_t^B\right) - \delta \right] K_t dt + \frac{\sigma}{\sqrt{2}} K_t \frac{dZ_t^a + dZ_t^b}{\sqrt{2}} \end{aligned} \quad (\text{P2})$$

Note that agents fully share the risks, the responses to Brownian shocks dZ_t^a and dZ_t^b will symmetric, and we can aggregate them into a single standard Brownian shock $dZ_t = (dZ_t^a + dZ_t^b)/\sqrt{2}$. The HJB equation for the planner's problem is:

$$\rho \bar{V}(K) = \max_{\zeta^A, \zeta^B \geq 0} \lambda^A U(\zeta^A K) + (1 - \lambda^A) U(\zeta^B K) + \bar{V}_K K \left[\Phi\left(\frac{a}{2} - \zeta^A - \zeta^B\right) - \delta \right] + \frac{1}{2} \bar{V}_{KK} K^2 \frac{\sigma^2}{2} \quad (\text{P3})$$

Solving for equilibrium total consumption rate ζ^* : We look at, in turn, the CRRA utility case and the log utility case.

Case I (CRRA Utility): Look for solutions of the form: $\bar{V}(K) = A \frac{K^{1-\gamma}}{1-\gamma}$ to (P3). Then $\bar{V}_K K = AK^{1-\gamma}$ and $\bar{V}_{KK} K^2 = -\gamma AK^{1-\gamma}$. Plug into (P3), HJB under this conjecture is:

$$\rho A \frac{K^{1-\gamma}}{1-\gamma} = \max_{\zeta^A, \zeta^B \geq 0} \left\{ \lambda^A \frac{(\zeta^A)^{1-\gamma}}{1-\gamma} + (1 - \lambda^A) \frac{(\zeta^B)^{1-\gamma}}{1-\gamma} + A \left[\Phi\left(\frac{a}{2} - \zeta^A - \zeta^B\right) - \delta \right] - \frac{\gamma A \sigma^2}{4} \right\} K^{1-\gamma} \quad (\text{P3}')$$

The terms in the bracket does not depend on K , which verifies the function form for \bar{V} . The first order conditions are:

$$\begin{aligned} (\lambda^A)^{\frac{-1}{\gamma}} (\zeta^A) &= \left[A \Phi' \left(\frac{a}{2} - \zeta^A - \zeta^B \right) \right]^{\frac{-1}{\gamma}}, \\ (1 - \lambda^A)^{\frac{-1}{\gamma}} (\zeta^B) &= \left[A \Phi' \left(\frac{a}{2} - \zeta^A - \zeta^B \right) \right]^{\frac{-1}{\gamma}}. \end{aligned}$$

Let total consumption rate be $\zeta^* = \zeta^A + \zeta^B$. Then:

$$\zeta^A = \frac{(\lambda^A)^{\frac{1}{\gamma}}}{(\lambda^A)^{\frac{1}{\gamma}} + (1 - \lambda^A)^{\frac{1}{\gamma}}} \cdot \zeta^*, \quad \text{and} \quad \zeta^B = \frac{(1 - \lambda^A)^{\frac{1}{\gamma}}}{(\lambda^A)^{\frac{1}{\gamma}} + (1 - \lambda^A)^{\frac{1}{\gamma}}} \cdot \zeta^*$$

The FOC becomes:

$$(\zeta^*)^{-\gamma} = \left[(\lambda^A)^{\frac{1}{\gamma}} + (1 - \lambda^A)^{\frac{1}{\gamma}} \right]^{-\gamma} \left[A\Phi' \left(\frac{a}{2} - \zeta^* \right) \right] \quad [\text{FOC}]$$

which involves A and ζ^* . For given constant A , LHS of the above equation is decreasing in the total consumption rate ζ^* and RHS is increasing in ζ^* .

To find both A and ζ^* we need one more equation. Plug $\lambda^A(\zeta^A)^{-\gamma} = A\Phi'$ and $(1 - \lambda^A)(\zeta^B)^{-\gamma} = A\Phi'$ into the HJB equation (P3') to find:

$$\frac{\rho A}{1 - \gamma} = \frac{\zeta^*}{1 - \gamma} A\Phi' \left(\frac{a}{2} - \zeta^* \right) + A \left[\Phi \left(\frac{a}{2} - \zeta^* \right) - \delta \right] - \frac{\gamma \sigma^2 A}{4} \quad [\text{HJB}']$$

The A 's cancel out and the [HJB'] becomes a single variate equation in ζ^* , [HJB'']:

$$\frac{\rho}{1 - \gamma} = \frac{\zeta^*}{1 - \gamma} \Phi' \left(\frac{a}{2} - \zeta^* \right) + \left[\Phi \left(\frac{a}{2} - \zeta^* \right) - \delta \right] - \frac{\gamma \sigma^2}{4} \quad [\text{HJB}'']$$

Importantly, the total consumption rate ζ^* does not depend on K_t . For appropriate functional form and parameters we can solve out ζ^* , the optimal total consumption rate, using [HJB''].

Case II (Log Utility): Look for solutions of the form: $\bar{V}(K) = A \log(K) + B$. Then $\bar{V}_K K = A$ and $\bar{V}_{KK} K^2 = -A$. First order conditions for the HJB equation (P3) are:

$$\begin{aligned}\frac{\lambda^A}{\zeta^A} &= A\Phi'\left(\frac{a}{2} - \zeta_t^A - \zeta_t^B\right), \\ \frac{1 - \lambda^A}{\zeta^B} &= A\Phi'\left(\frac{a}{2} - \zeta_t^A - \zeta_t^B\right).\end{aligned}$$

Under log utility, consumption shares are proportional to welfare weights ($\frac{\zeta^A}{\zeta^B} = \frac{\lambda^A}{1-\lambda^A}$). Furthermore, since the ζ 's does not depend on K (scale invariance), coefficient A equals ρ^{-1} and we verify the function form assumed on \bar{V} .

Let $\zeta \equiv \zeta^A + \zeta^B$, the optimal consumption rate ζ^* is then pinned down by:

$$(\zeta^*)^{-1} = \frac{1}{\rho}\Phi'\left(\frac{a}{2} - \zeta^*\right) \quad [\text{FOC}(\zeta^*)]$$

where the LHS is marginal utility in rates and the RHS is the marginal efficiency of capital investment times the marginal value of capital (in rates). Importantly, ζ^* does not depend on K_t . The LHS is decreasing in ζ^* from $+\infty$ to 0 and the RHS is increasing in ζ^* from $\Phi'(\frac{a}{2}) > 0$ to $\Phi'(-\infty)$. There uniquely exists a ζ^* that solves the planner's problem. The individual consumption rates are $\zeta^{A*} = \lambda^A\zeta^*$ and $\zeta^{B*} = (1 - \lambda^A)\zeta^*$. The value function \bar{V} for the social planner's problem is:

$$\bar{V}(K) = \rho^{-1} [\log(\zeta^*K) + \lambda^A \log(\lambda^A) + (1 - \lambda^A) \log(1 - \lambda^A)] + \rho^{-2} \left[\Phi\left(\frac{a}{2} - \zeta^*\right) - \delta \right] - \frac{\sigma^2}{4\rho^2}.$$

Decentralization: Representative Agent Economy

The original problem (P1) and the planner's problem (P2) is equivalent to a representative agent with utility function \tilde{U} defined by

$$\tilde{U}(\zeta K) \equiv \max_{\zeta^A + \zeta^B \leq \zeta} U(\zeta^A K) + (1 - \lambda^A)U(\zeta^B K)$$

facing the following problem:

$$\begin{aligned} \tilde{V}^R(K_0) &= \max_{\zeta \geq 0} E_0 \left[\int_0^\infty e^{-\rho t} \left[\tilde{U}(\zeta_t K_t) \right] dt \right] \\ \text{subject to: } dK_t &= \left[\Phi\left(\frac{a}{2} - \zeta_t\right) - \delta \right] K_t dt + \frac{\sigma}{\sqrt{2}} K_t dZ_t \end{aligned} \quad (\text{P4})$$

And the equilibrium stochastic discount factor is given by:

$$m_t = \frac{e^{-\rho t} \tilde{U}'(\zeta^*(K_t) K_t)}{\tilde{U}'(\zeta^*(K_0) K_0)}$$

We will be able to price assets if we know the representative agent's utility function \tilde{U} and the equilibrium total consumption rate $\zeta^*(K)$. Importantly, for both CRRA and log utility, the representative agent inherits the utility function of the agents (i.e. CRRA(γ) for CRRA(γ) and log for log)) and the optimal total consumption rate is invariant to K , i.e. $\zeta^*(K) = \zeta^*$. The stochastic discount factor m_t can then be simplified:

$$m_t = e^{-\rho t} \left(\frac{K_0}{K_t} \right)^\gamma \quad (\gamma > 1 \rightarrow \text{CRRA utility}; \gamma = 1 \rightarrow \text{log utility})$$

By Ito's Lemma, the SDF m_t evolves according to the law of motion:

$$\frac{dm_t}{m_t} = \left\{ -\rho - \gamma \left[\Phi\left(\frac{a}{2} - \zeta^*\right) - \delta \right] + \frac{\gamma(\gamma + 1)\sigma^2}{4} \right\} dt - \frac{\gamma\sigma}{\sqrt{2}} dZ_t$$

The risk-free interest rate r is given by:

$$r = -E \left[\frac{dm_t}{m_t dt} \right] = \rho + \gamma \left[\Phi\left(\frac{a}{2} - \zeta^*\right) - \delta \right] - \frac{\gamma(\gamma + 1)\sigma^2}{4}$$

And the price of capital should be constant since the economy is scale invariant and in first best. Let $dq_t/q_t = 0$. Then

$$\frac{dq_t K_t}{q_t K_t} = \left[\Phi\left(\frac{a}{2} - \zeta_t\right) - \delta \right] dt + \frac{\sigma}{\sqrt{2}} dZ_t$$

The return process of investing in capital is:

$$dr_t^K = \frac{\zeta^*}{q_t} dt + \left[\Phi\left(\frac{a}{2} - \zeta_t\right) - \delta \right] dt + \frac{\sigma}{\sqrt{2}} dZ_t$$

The following asset pricing relationship holds for capital: $E\left[\frac{dr_t^K m_t}{m_t d_t}\right] = 0$. Using Ito's Lemma, we get:

$$0 = \mu_t^m + \mu_t^{r^K} + \sigma_t^{r^K} \sigma_t^m$$

$$\Rightarrow r = \frac{\zeta^*}{q_t} + \left[\Phi\left(\frac{a}{2} - \zeta_t\right) - \delta \right] + \frac{\sigma}{\sqrt{2}} * \left(-\frac{\gamma\sigma}{\sqrt{2}} \right)$$

Hence the first best price of capital q_t is:

$$q_t = \frac{\zeta^*}{r + \frac{\gamma}{2}\sigma^2 - \left[\Phi\left(\frac{a}{2} - \zeta^*\right) - \delta \right]}$$

It is worth noting that the price of capital q_t in the first best follows Gordon's growth formula, where r is the interest rate, $\frac{\gamma}{2}\sigma^2$ is a risk premium, and $\left[\Phi\left(\frac{a}{2} - \zeta^*\right) - \delta \right]$ is the growth rate.

For a specific investment technology $\Phi(i) = \frac{1}{\kappa} (\sqrt{1 + 2\kappa i} - 1)$, parameter values $a = 0.14$, $\delta = 0.05$, $\kappa = 2$, $\rho = 0.05$ and log utility, we calculate the first best interest rate and price of capital. Plug the specifications into $[\text{FOC}(\zeta^*)]$ in the log utility discussion above, we calculate the optimal total consumption rate to be $\zeta^* = 0.16$.⁷ The first best interest rate is $r = -0.1 - \sigma^2$.

B Proof of Proposition 2

Proof. Agents with logarithmic utility consume at rate of ρ times their net worth. Thus, equations (13) and (15) follow from the market-clearing condition for output.

⁷ $x^{-1} = \rho^{-1}\Phi'\left(\frac{a}{2} - x\right)$ and $\Phi'(\iota) = [2\kappa\sqrt{1 + 2\kappa\iota}]^{-1} \Rightarrow \frac{x^2}{4\rho^2\kappa^2} + 2\kappa x - a\kappa - 1 = 0$. With given parameters $\Rightarrow 25x^2 + 4x - 1.28 = 0$. The positive root $x = 0.16$ is the optimal total consumption rate ζ^* .

Next, from the pricing conditions for capital held by agents B (see (9) and (10)),

$$\frac{aP^b - \underline{a}P^a}{q_t} + \sigma\sigma^{qb} - \sigma\sigma^{qa} - \gamma\sigma_t^{Bb}\sigma + \gamma\sigma_t^{Ba}\sigma = 0. \quad (19)$$

Since consumption is proportional to net worth, the volatility of consumption equals to that of net worth. The net worth of agents B follows

$$\frac{dN_t^B}{N_t^B} = \frac{\psi_t^{Bb}}{1 - \eta_t} dr_t^{Bb} + \frac{1 - \eta_t - \psi_t^{Bb}}{1 - \eta_t} dr_t^{Ba} - \frac{C_t^B}{N_t^B} dt.$$

Therefore, in the region $[0, 1 - \eta^\psi)$,

$$\sigma_t^{Bb} = \frac{\psi_t^{Bb}}{1 - \eta_t} \sigma + \sigma_t^{qb} \quad \text{and} \quad \sigma_t^{Ba} = \frac{1 - \eta_t - \psi_t^{Bb}}{1 - \eta_t} \sigma + \sigma_t^{qa}. \quad (20)$$

Combining this with (21) and setting $\gamma = 1$, we obtain

$$\frac{aP^b - \underline{a}P^a}{q_t} + \frac{1 - \eta_t - 2\psi_t^{Bb}}{1 - \eta_t} \sigma^2 = 0, \quad (21)$$

which is equivalent to (14).

Finally, we need to derive the law of motion of η_t . The net worth of agents A on $[0, \eta^b)$ follows

$$\frac{dN_t^A}{N_t^A} = \underbrace{\left(\frac{aP_t^a - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^{qa} \right)}_{dr_t^{Aa}} dt + (\sigma + \sigma_t^{qa}) dZ_t^a + \sigma_t^{qb} dZ_t^b - \rho dt.$$

The aggregate net worth of all agents follows

$$\begin{aligned} & (\mu_t^q + \Phi(\iota_t) - \delta + (1 - \psi^{Bb})\sigma\sigma^{qa} + \psi^{Bb}\sigma\sigma^{qb}) dt \\ & + ((1 - \psi^{Bb})\sigma + \sigma_t^{qa}) dZ_t^a + (\psi^{Bb}\sigma + \sigma_t^{qb}) dZ_t^b. \end{aligned}$$

Therefore, using Ito's lemma, the volatility of η is given by $\sigma_t^{\eta a} = \psi^{Bb}\sigma$, $\sigma_t^{\eta b} = -\psi^{Bb}\sigma$.

Likewise, the drift of η_t is

$$\begin{aligned} & \frac{aP_t^a - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta + \sigma\sigma_t^{qa} - \rho - (\mu_t^q + \Phi(\iota_t) - \delta + (1 - \psi^{Bb})\sigma\sigma^{qa} + \psi^{Bb}\sigma\sigma^{qb}) \\ & - (\sigma + \sigma_t^{qa})((1 - \psi^{Bb})\sigma + \sigma_t^{qa}) - \sigma_t^{qb}(\psi^{Bb}\sigma + \sigma_t^{qb}) + ((1 - \psi^{Bb})\sigma + \sigma_t^{qa})^2 + (\psi^{Bb}\sigma + \sigma_t^{qb})^2, \end{aligned}$$

which leads to (16) after simplifications. ■

C Proof of Proposition 3

Proof. Combining equations (9) with (11), we find that the return that agents A earn from technology a , over the risk-free rate, is

$$(\gamma\sigma_t^{Aa}(\sigma^a + \sigma_t^{qa}) + \gamma\sigma_t^{Ab}\sigma_t^{qb}) dt + (\sigma^a + \sigma_t^{qa}) dZ_t^a + \sigma_t^{qb} dZ_t^b.$$

Combining equations (10) with (11), the return from technology b over the risk-free rate, if $\psi_t^{Ab} > 0$, is

$$(\gamma\sigma_t^{Aa}\sigma_t^{qa} + \gamma\sigma_t^{Ab}(\sigma^b + \sigma_t^{qb})) dt + \sigma_t^{qa} dZ_t^a + (\sigma^b + \sigma_t^{qb}) dZ_t^b.$$

With portfolio weights ψ_t^{Aa}/η_t and ψ_t^{Ab}/η_t on the two technologies, the law of motion of the net worth of agents A is

$$\begin{aligned} \frac{dN_t}{N_t} &= \gamma \left(\frac{\psi_t^{Aa}}{\eta_t} (\sigma_t^{Aa}(\sigma^a + \sigma_t^{qa}) + \sigma_t^{Ab}\sigma_t^{qb}) + \frac{\psi_t^{Ab}}{\eta_t} (\sigma_t^{Aa}\sigma_t^{qa} + \sigma_t^{Ab}(\sigma^b + \sigma_t^{qb})) \right) dt \\ &+ dr_t^F - \frac{C_t}{N_t} dt + \left(\frac{\psi_t^{Aa}}{\eta_t} \sigma^a + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qa} \right) dZ_t^a + \left(\frac{\psi_t^{Ab}}{\eta_t} \sigma^b + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qb} \right) dZ_t^b. \end{aligned}$$

The law of motion of aggregate wealth can be found by computing the return on the aggregate portfolio of capital, and subtracting the dividend yield, i.e.

$$\frac{d(q_t K_t)}{(q_t K_t)} = \gamma\psi_t^{Aa} \left((\sigma^a + \sigma_t^{qa})\sigma_t^{Aa} + \sigma_t^{qb}\sigma_t^{Ab} \right) dt + \gamma\psi_t^{Ab} \left(\sigma_t^{qa}\sigma_t^{Aa} + (\sigma^b + \sigma_t^{qb})\sigma_t^{Ab} \right) dt$$

$$\begin{aligned}
& +\gamma\psi_t^{Ba} \left((\sigma^a + \sigma_t^{qa})\sigma_t^{Ba} + \sigma_t^{qb}\sigma_t^{Bb} \right) dt + \gamma\psi_t^{Bb} \left(\sigma_t^{qa}\sigma_t^{Ba} + (\sigma^b + \sigma_t^{qb})\sigma_t^{Bb} \right) dt + dr_t^F \\
& - \frac{Y_t - \iota_t K_t}{q_t K_t} dt + ((\psi_t^{Aa} + \psi_t^{Ba})\sigma^a + \sigma_t^{qa}) dZ_t^a + ((\psi_t^{Ab} + \psi_t^{Bb})\sigma^b + \sigma_t^{qb}) dZ_t^b.
\end{aligned}$$

It may appear strange that we derived this expression by subtracting dividend yield from the return on the world portfolio, instead of by multiplying the laws of motion of q and K . The benefit of this approach is that it allows us to express the law of motion of η_t without using μ_t^q and r_t^F . Thus, the law of motion of η that we obtain in the end can be computed purely from the first derivatives of q , C^A and C^B (without second derivatives).

Using Ito's lemma,

$$\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \frac{dN_t}{N_t} - \frac{d(q_t K_t)}{(q_t K_t)} - \left(\frac{\psi_t^{Aa}}{\eta_t} \sigma^a + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qa} \right) ((\psi_t^{Aa} + \psi_t^{Ba})\sigma^a + \sigma_t^{qa}) dt \\
&\quad - \left(\frac{\psi_t^{Ab}}{\eta_t} \sigma^b + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qb} \right) ((\psi_t^{Ab} + \psi_t^{Bb})\sigma^b + \sigma_t^{qb}) dt \\
&\quad + ((\psi_t^{Aa} + \psi_t^{Ba})\sigma^a + \sigma_t^{qa})^2 dt + ((\psi_t^{Ab} + \psi_t^{Bb})\sigma^b + \sigma_t^{qb})^2 dt = \\
&\quad \gamma \frac{1 - \eta_t}{\eta_t} \left(\psi_t^{Aa} (\sigma_t^{Aa} (\sigma^a + \sigma_t^{qa}) + \sigma_t^{Ab} \sigma_t^{qb}) + \psi_t^{Ab} (\sigma_t^{Aa} \sigma_t^{qa} + \sigma_t^{Ab} (\sigma^b + \sigma_t^{qb})) \right) dt \\
&\quad - \gamma \psi_t^{Ba} \left((\sigma^a + \sigma_t^{qa})\sigma_t^{Ba} + \sigma_t^{qb}\sigma_t^{Bb} \right) dt - \gamma \psi_t^{Bb} \left(\sigma_t^{qa}\sigma_t^{Ba} + (\sigma^b + \sigma_t^{qb})\sigma_t^{Bb} \right) dt + \\
&\quad \frac{Y_t - \iota_t K_t}{q_t K_t} dt - \frac{C_t}{N_t} dt - \sigma_t^{\eta a} ((\psi_t^{Aa} + \psi_t^{Ba})\sigma^a + \sigma_t^{qa}) dt + \sigma_t^{\eta b} ((\psi_t^{Ab} + \psi_t^{Bb})\sigma^b + \sigma_t^{qb}) dt \\
&\quad + \underbrace{\left(\frac{\psi_t^{Aa}}{\eta_t} \sigma^a + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qa} - (\psi_t^{Aa} + \psi_t^{Ba})\sigma^a - \sigma_t^{qa} \right)}_{\sigma_t^{\eta a}} dZ_t^a
\end{aligned}$$

$$+ \underbrace{\left(\frac{\psi_t^{Ab}}{\eta_t} \sigma^b + \frac{\psi_t^{Aa} + \psi_t^{Ab}}{\eta_t} \sigma_t^{qb} - (\psi_t^{Ab} + \psi_t^{Bb}) \sigma^b - \sigma_t^{qb} \right)}_{\sigma_t^{\eta^b}} dZ_t^b.$$

We obtain (17). ■