When queueing is better than push and shove*

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Abstract

We address the scheduling problem of reordering an existing queue into its efficient order through trade. To that end, we consider individually rational and balanced budget direct and indirect mechanisms. We show that this class of mechanisms allows us to form efficient queues provided that existing property rights for the service are small enough to enable trade between the agents. In particular, we show on the one hand that no queue under a fully deterministic service schedule such as first-come, first-serve can be dissolved efficiently and meet our requirements. If, on the other hand, the alternative is full service anarchy then every existing queue can be transformed into its efficient order. (JEL *C72, D44, D82.* Keywords: *Scheduling, Queueing, Mechanism design.*)

1 Introduction

We analyse the problem of organising efficient sequential access of a set of agents to some service. All agents value the service equally but have privately known waiting costs. Hence there is the potential for an improvement in efficiency relative to an existing waiting queue through the trade of service rights. Efficient access is to be organised only among the agents themselves, without payments from or to outsiders. We show that for fully deterministic queues where agents are issued with non-probabilistic slot tickets, it is impossible to achieve an efficient order using a mechanism from the above class. If, however, agents initially face service anarchy in the sense of equal probability of service in each slot, then they can mutually agree to implement the efficient order.

An example of our setup arises with the potential short-term, dynamic trade of airport landing or takeoff slots.[1] A short-term slot trading mechanism—for some fixed period of takeoff or landing activity in advance—is a scheduling problem since the sets of arriving and departing airplanes are known for the period considered. Ball, Donohue, and Hoffman (2006, p533) argue for a near-real-time market that allows for the trading of slots: "A key property of these slot-trading markets is that each airline is potentially both a buyer and seller. In fact, the natural

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[1] The US AIR-21 Bill prescribes the full deregulation of slot controls at the US High Density Rule airports New York John F Kennedy and LaGuardia, Chicago O'Hare and Washington Reagan National. The Federal Aviation Administration's slot auction plans have been challenged, however, and the US Circuit Court of Appeals for the District of Columbia has suspended the proposed slot auctions pending its review in 2009.

extension of the current exchange system suggests simply adding the possibility of side payments to the current trades." Vindicating our balanced budget condition, they point out that the authorities running airports are "almost always public agencies [and] typically restricted in that their charges for services can only achieve cost recovery." Airlines buy slots on the basis of their long-term flight schedules and a short-term market only becomes potentially beneficial as new information arrives. Individual rationality captures the aspect that the operators will only be active in this short-term market if this is to their advantage.

Setting ethical considerations aside, another example is the waiting system of the British National Health Service (NHS). There, patients for certain procedures are put on a waiting list with the ranking being based on their doctors' diagnoses. As a result, patients with the same diagnosis class are treated first-come, first-serve but may have differing and privately known waiting costs. 'Private' patients often use the same facilities, doctors and staff, but are not subject to the same schedule. They are typically treated without significant waiting and their payments are made to the service provider. If trade between queue-positions in a single queue were possible, the payments made by these private patients would accrue to the other patients whose wait is prolonged through the speedier servicing of the private patients. Individual rationality of the mechanism is ensured through the universality principle of the NHS: everyone is entitled to its services and may or may not accept the offered payments for switching positions.[2]

Our contribution is to show that two important theoretical results can be extended to provide novel and powerful implications in the context of queueing. These are Myerson and Satterthwaite (1983), on the one hand, who show the inexistence of efficient and individually rational trading mechanisms for a wide class of incomplete information problems with asymmetric ownership distributions. On the other hand, Cramton, Gibbons, and Klemperer (1987) derive the contrary result that there can be efficient trade among a group of agents provided that initial ownership is equally distributed. We show that the analogue of the former is any deterministic rule or, in particular, the first-come, first-serve (fcfs) schedule.[3] It cannot be efficiently rescheduled. The analogue of the latter turns out to be a random schedule. This type of queue can be rescheduled efficiently. Actually, the problem of efficiently reorganising a two player deterministic queue is equivalent to the Myerson-Satterthwaite environment of efficient trade under incomplete information. The generalisation to more than two players (or more than one object) in the Myerson-Satterthwaite trading setup is typically done by replicating both sides of the markets. This differs substantially from our environment as each player (other than those holding the first and last fcfs positions) is both a buyer and seller of service rights

[2] Further applications are the joint scheduling of jobs by different profit centres on a corporate shop floor, farm machinery co-operatives whose individual members' needs for jointly owned machinery may arise simultaneously, the scheduling of trains, ships' servicing at sea ports, the general "control of vehicular traffic congestion" (Naor, 1969), and individual access to pooled corporate or research facilities.

[3] By a deterministic rule we mean a scheduling rule where an agent is served with probability one at a particular slot. Throughout the paper we use the fcfs schedule as representative for any such deterministic rule. Stochastic schedules do not assign full probability mass to a single slot. The random schedule is such a rule which assigns equal probability of being served at any slot to all agents.

when moving to the efficient slot.

Two frequently applied service schedules are the fcfs and the random schedules. Since both these procedures are inefficient, we analyse whether there exists a game which implements the efficient allocation and improves the utilities of all players in the queue without making a budget deficit. While we show that a game with these properties indeed exists for the random schedule, it does not exist for the fcfs order or any other fully deterministic rule.

Our general result is obtained for a direct mechanism. To illustrate its workings we analyse an indirect bidding game in section 4 which can be applied to the examples discussed above. This game is reminiscent of Engelbrecht-Wiggans (1994) who subdivides a single winning bid among all bidders in a single-unit auction game which he motivates with the study of bidding rings. The rules of our indirect mechanism are such that each of the $n > 2$ winning bids is subdivided among all $n - 1$ players not obtaining that object. We show that this game possesses an equilibrium implementing the efficient allocation and discuss the intuition of this result. Finally, we show that indirect mechanisms of the class analysed are individually rational with respect to a random queue.

Related literature

Most of the existing non-cooperative literature analysing scheduling problems is based on the *Vickrey-Clarke-Groves* (VCG) mechanism and typically ignores existing service rights. However, individual rationality relative to some existing mechanism must be scrutinised in order to ensure that no agent is made worse off by adopting a new mechanism. Therefore our analysis examines whether efficient rescheduling is possible when agents are individually rational with respect to the status quo.

The path breaking work on scheduling problems based on the VCG mechanism is Dolan (1978). Suijs (1996) assumes linear cost (as we do) and shows that a VCG mechanism implements the efficient order in dominant strategies and balances the budget. He deviates from standard assumptions by examining individual rationality with respect to not getting the service at all and gives an example showing that sequencing an initial fcfs queue violates individual rationality. Mitra (2002) shows that linear cost functions are the only cost functions which can lead to an efficient allocation in dominant strategies if budget balancedness is required. Hain and Mitra (2004) allow for processing time to be private information. They identify the class of mechanisms which lead to an efficient allocation and balance the budget in ex-post equilibrium. We stress that these two contributions analyse VCG mechanisms and do not impose individual rationality with respect to an existing mechanism.

The queueing literature studies the aspect of our problem that arriving customers can gain priority over others through a single one-off payment to the service provider under the heading of 'priority pricing.' Hassin and Haviv (2002) survey the recent queueing literature including models where the queueing agents offer payments to the service provider. Mitra (2001) uses a mechanism design approach to identify the cost functions for which queues can be efficiently reorganised in dominant strategies while balancing the budget. He further derives a subset of individually rational cost functions where non-participation means obtaining no service at all.

From the point of view of general mechanism design, our study can be seen as an extension of Cramton, Gibbons, and Klemperer (1987) to more than one object. Their contribution is to show that the Myerson-Satterthwaite impossibility can be overcome when leaving ownership of a single object ex-ante undecided. We generalise this insight to multiple objects interpreted as service positions in a queue with the attached service probabilities reflecting ownership concentration. Moreover, we provide an impossibility result for deterministic queues.

Krishna and Perry (1997) and, similarly, Williams (1999), analyse general mechanism design problems. In particular, they show that there exists an efficient and individually rational mechanism that balances the budget if and only if the generalised VCG mechanism they define runs an expected surplus. The generality of their analysis, however, does not allow for specific insights to be developed into the problem of the rescheduling of queues. For example, our more specific approach allows the development of an indirect game which provides intuition on the economics at work. Moreover, the conditions derived on the direct mechanism facilitate new insights into queueing to be developed.

The following section introduces the model. In section 3 we develop the direct revelation game and provide illustrative three players examples. In section 4 we construct an indirect game implementing the efficient schedule. All proofs and details are contained in the appendix.

2 The model

There is a set $\mathcal{N}$ of $n > 1$ players willing to get some specific service valued at V. Although the service is valuable, every player incurs a cost of waiting to get the service. More precisely, for $i \in \mathcal{N}$, we assume that player i's utility from getting the service at the k^{th} period is $V - k\theta_i - p$ where p is a payment by agent i and θ_i is waiting cost per unit of time.[4] The server can serve only one player at each point in time. Waiting cost $\theta_i \in \Theta_i = [0, 1]$ is private information and independently distributed with positive density f and distribution function F. Finally, we assume that the processing time of any player is the same and normalised to one period.

The mechanism designer wishes to implement the efficient order of service, which coincides with a decreasing ranking of the agents' waiting cost. This maximises the aggregated expected utility of the players. By the *revelation principle* we may restrict attention to direct revelation mechanisms, where the players only reveal their private information to the designer. Denote by $\Theta = [0, 1]^n$ the type space and by θ any element of Θ. The mechanism M has to specify the payment that each player should make and the (possibly stochastic) order of getting the service. In the general direct mechanism, both this payment and order may depend on the initial allocation specified by $\sigma_i \in I_i \equiv [0, 1]^n$, with $\sum_j \sigma_{ij} = \sum_i \sigma_{ij} = 1$, where σ_{ij} specifies the probability that agent i is served at the j^{th} period. We denote by σ the full vector of $\langle \sigma_i \rangle_{i=1}^n$ and by $I = [0, 1]^{n \times n}$ the space of all initial allocations. Therefore a direct revelation mechanism is a vector of payments $p^M = \left\langle p_i^M \right\rangle_{i=1}^n$ and the order $\sigma^M = \left\langle \sigma_{ij}^M \right\rangle_{i,j=1}^n$, where $p_i^M : \Theta \times I \to \mathbb{R}$

[4] As customary in the literature, we do not consider discounting of payments.

is player i's payment and, for $1 \leq i, j \leq n$, $\sigma_{ij}^M : \Theta \times I \to [0,1]$. Consequently we again have $\sum_i \sigma_{ij}^M(\theta, \sigma) = 1$ for each j and $\sum_j \sigma_{ij}^M(\theta, \sigma) = 1$ for each i. Therefore, if all players report their observed signals θ_{-i} truthfully, the expected utility of player i observing signal θ_i is

$$U_i(\theta_i, \sigma) = V - \mathbb{E}_{\theta_{-i}} \left[\sum_{k=1}^{n} \sigma_{ik}^M(\theta, \sigma) k \theta_i + p_i^M(\theta, \sigma) \right]$$

where $\theta = (\theta_i, \theta_{-i})$. Denote by $W_i^M(\theta_i, \sigma)$ and $P_i^M(\theta_i, \sigma)$ the expected waiting time and the payment by player i if θ_i is the reported delay cost. That is,

$$P_i^M(\theta_i, \sigma) = \mathbb{E}_{\theta_{-i}} \left[p_i^M(\theta, \sigma) \right], \quad W_i^M(\theta_i, \sigma) = \sum_{k=1}^{n} k \, \mathbb{E}_{\theta_{-i}} \left[\sigma_{ik}^M(\theta, \sigma) \right].$$

We define individual rationality wrt some initial allocation Z as the requirement for the target mechanism M to give at least the same utility in expected terms, that is, for any $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$,

$$V - W_i^M(\theta_i, \sigma)\theta_i - P_i^M(\theta_i, \sigma) \geq V - W_i^Z(\theta_i, \sigma)\theta_i.$$

Moreover, M is incentive compatible if, for any i and any $\theta_i, \hat{\theta}_i \in \Theta_i$, it is true that

$$-W_i^M(\theta_i, \sigma)\theta_i - P_i^M(\theta_i, \sigma) \geq -W_i^M(\hat{\theta}_i, \sigma)\theta_i - P_i^M(\hat{\theta}_i, \sigma).$$

Three possible service schedules are of interest to us: an initially stochastic order, an initially deterministic schedule determined through something other than the private information and the efficient order in the target mechanism.

1. *Random order.* Each player is at any position with equal probability. That is,

$$\sigma_{ik}^{\text{ran}} = \frac{1}{n} \text{ for any } i, k, \theta.$$

2. *First-come, first-serve (fcfs) order.* In this case, players are served according to their arrival time or some other deterministic, exogenous parameter unrelated to waiting cost. That is, for any player i there exists a unique position l such that

$$\sigma_{ik}^{\text{fcfs}} = \begin{cases} 1 & \text{if } l = k \\ 0 & \text{otherwise.} \end{cases}$$

3. *Efficient order.* Players are served based on decreasing waiting cost, ie. in $M = \text{ef}$,

$$\sigma_{ik}^{\text{ef}}(\theta, \sigma) = \begin{cases} 1 & \text{if } |\{j : \theta_j > \theta_i\}| = k - 1 \text{ and } |\{j \neq i : \theta_j = \theta_i\}| = 0 \\ {}^1\!/_m & \text{if } |\{j : \theta_j > \theta_i\}| = l \text{ and } |\{j \neq i : \theta_j = \theta_i\}| = m \neq 0, \text{ where } l + m \geq k > l \\ 0 & \text{otherwise,} \end{cases}$$

where $|S|$ is the number of elements of set S.

In the following we deal with the question of which kind of schedule can be improved upon in the mutual interest. Hence we analyse the question whether there exists a game that induces the efficient allocation, provides all types of all players with expected utilities higher then the one obtained in the random or fcfs order while balancing the budget ex post.

3 Direct mechanism

The first lemmata of our analysis of the direct mechanism give standard properties of any incentive compatible, individually rational and balanced budget mechanism. In particular, the single-slot version of our mechanism studied by Cramton, Gibbons, and Klemperer (1987) provides a version of these lemmata. As their proofs are standard, they are omitted. The first lemma specifies necessary and sufficient conditions for the mechanism to be incentive compatible. In particular, it says that in any incentive compatible mechanism, increasing the delay cost should lead to earlier service and higher payment of that player. This is similar to a standard result in auction theory—Myerson (1981), among others—which says that in any incentive compatible mechanism, the probability to get the object increases with a player's valuation.

Lemma 1. *The mechanism $M = \langle p^M, \sigma^M \rangle$ is incentive compatible iff, for every $i \in \mathcal{N}$ and all $\overline{\theta}, \underline{\theta} \in [0,1]$, W_i^M is decreasing in θ_i and*

$$P_i^M(\underline{\theta}, \sigma) - P_i^M(\overline{\theta}, \sigma) = \int_{\underline{\theta}}^{\overline{\theta}} s dW_i^M(s, \sigma). \tag{1}$$

The players prefer to adopt any new mechanism if it provides them with higher utility then the original mechanism. Hence we check whether our proposed mechanism is individually rational when the outside option is either the random schedule or the fcfs order.[5] The next lemma specifies the type of each player who gains least among all possible types of that player by moving to the efficient mechanism $\langle p^M, \sigma^M \rangle$. In the following analysis, we call this type the 'worst-off' type $\theta^*(Z)$ relative to the status quo Z. As long as the participation of this type can be ensured, all other types will voluntarily trade their positions as well. The lemma says that the net utility is minimised for the type of player who on average stays at the same position.

Lemma 2. *Given an incentive compatible mechanism $M = \langle p^M, \sigma^M \rangle$, player i's net utility wrt mechanism $Z \in \{ran, fcfs\}$ is minimised at*

$$\theta_i^*(Z) = \frac{1}{2}\left[\inf \Theta_i^*(Z) + \sup \Theta_i^*(Z)\right] \in [0,1] \tag{2}$$

where $\Theta_i^*(Z) = \left\{\theta_i | W_i^M(\theta_j, \sigma) < W_i^Z(\theta_i, \sigma), \forall \theta_j < \theta_i; W_i^M(\theta_k, \sigma) > W_i^Z(\theta_i), \forall \theta_k > \theta_i\right\}$.

[5] As indicated in the introductory discussion and formalised in corollary 1 below, we do not limit our analysis to the random and fcfs orders. These are merely the two extremes which can govern the initial allocation.

The idea of the proof uses the observation that the players' additional utility from participating in the efficient mechanism is convex in their type.

Lemma 3. *In the efficient schedule $M = ef$, type θ_i's expected waiting time is*

$$W_i^{ef}(\theta_i, \sigma) = n + (1-n)F(\theta_i). \tag{3}$$

Without loss of generality and for notational simplicity, we will from now on assume that player i is served in position i if he is in the fcfs schedule. As player i's waiting time under the fcfs schedule is then just i, the worst-off type $\theta_i^*(\text{fcfs})$ is given by

$$n + (1-n)F(\theta_i^*(\text{fcfs})) = i \quad \text{or} \quad F(\theta_i^*(\text{fcfs})) = \frac{i-n}{1-n} \quad \text{and} \quad \theta_i^*(\text{fcfs}) = F^{-1}\left(\frac{n-i}{n-1}\right).$$

Waiting time in the random queue is

$$\frac{1}{n}1 + \frac{1}{n}2 + \cdots + \frac{1}{n}n = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\left(\frac{n^2+n}{2}\right) = \frac{n+1}{2}$$

and thus the worst-off type $\theta^*(\text{ran})$ solves

$$n + (1-n)F(\theta^*(\text{ran})) = \frac{n+1}{2} \quad \text{or} \quad \theta^*(\text{ran}) = F^{-1}({}^1\!/_2).$$

Next we derive a condition for an incentive compatible mechanism to be individually rational. The lemma says that since the worst-off type expects neither to sell nor buy a position, he (on average) cannot pay a positive transfer while reporting his type truthfully.

Lemma 4. *An incentive compatible mechanism $M = \langle p^M, \sigma^M \rangle$ is individually rational wrt mechanism $Z \in \{ran, fcfs\}$ iff, for all $i \in \mathcal{N}$ and $\theta_i^*(Z)$ as defined in lemma 2,*

$$P_i^M(\theta_i^*(Z), \sigma) \leq 0. \tag{4}$$

The proof is an immediate consequence of individual rationality and the specification of the worst-off type $\theta_i^*(Z)$. The next lemma specifies a condition for the budget to balance, ie. $\sum_i p_i^M(\theta, \sigma) = 0$, in the mechanism that satisfies incentive compatibility and individual rationality.

Lemma 5. *For any expected waiting time $W_i^M(\theta_i, \sigma)$ which is decreasing in θ_i for all $i \in N$, there exists a transfer function p^M such that $\langle p^M, \sigma^M \rangle$ is incentive compatible, individually rational wrt mechanism $Z \in \{ran, fcfs\}$ and budget balanced iff, for $\theta_i^*(Z)$ defined in lemma 2,*

$$\sum_{i=1}^{n}\left[\int_0^{\theta_i^*(Z)} sF(s)dW_i^M(s,\sigma) - \int_{\theta_i^*(Z)}^1 s(1-F(s))dW_i^M(s,\sigma)\right] \geq 0. \tag{5}$$

The proof uses the incentive compatibility condition obtained in lemma 1 and the implication from individual rationality of lemma 4. It shows that (5) is necessary for balancing the budget even ex-ante.

Now all results are in place to start the analysis of the efficient schedule. In the following we refer to the efficient queueing schedule as implementable with respect to any other discipline Z, if and only if there exists a mechanism $\langle p^M, \sigma^{\mathrm{ef}} \rangle$ which is incentive compatible, individually rational wrt Z and budget balanced. The following theorem pulls lemmata 3 and 5 together.

Theorem 1. *Efficient scheduling is implementable wrt schedule $Z \in \{ran, fcfs\}$ iff*

$$\sum_{i=1}^{n} \left[\int_0^{\theta_i^*(Z)} sF(s)f(s)ds - \int_{\theta_i^*(Z)}^{1} s(1-F(s))f(s)ds \right] \leq 0 \tag{6}$$

$$\textit{where} \quad \theta_k^*(Z) = \begin{cases} F^{-1}\left(\frac{n-k}{n-1}\right) & \textit{if} \quad Z = fcfs \\ F^{-1}\left({}^1\!/_2\right) & \textit{if} \quad Z = ran \end{cases}$$

and k is the position of player i in the fcfs schedule.

Notice that the worst possible type $\theta_k^*(\text{fcfs})$ depends on the initial position of the player in the fcfs queue. This implies that individual rationality has to be examined for each of the n slots individually, leading to n separate conditions for any deterministic schedule. The next proposition shows that an initially random schedule can always be rescheduled efficiently.

Proposition 2. *For any distribution of types F, the efficient scheduling is implementable wrt the random order.*

Although the first steps of our analysis of the direct mechanism above were similar to any other study of individually rational and balanced budget mechanisms, the next result is unique to our analysis and the queueing environment. In particular, while our implementability condition (6) is similar to condition (D) in Cramton, Gibbons, and Klemperer (1987), the agent's unit demand inherent in scheduling problems makes our setup incompatible with their analysis. The efficient allocation in their setup is simpler because the player with the highest valuation gets all available objects and no-one else gets anything. In our queueing problem, by contrast, we have to peel off the high-valuation players one after the other. As we show in the following proposition, no fully deterministic initial schedule such as fcfs can be rescheduled efficiently.

Proposition 3. *For any distribution of types F, the efficient scheduling is not implementable wrt first-come, first-serve order.*

The idea for proving this proposition is to consider an auxiliary distribution of types F^* which removes all uncertainty about types by concentrating all probability mass on a single type. For this auxiliary distribution F^* we show that (6) holds with equality. We proceed to show that any other distribution putting less weight on a single type than F^* will violate (6). In particular, we show that by decomposing any continuous distribution F into mass points, the opposite of the implementability condition (6) holds. This is done in two steps: Step 1 decomposes the original type distribution F into a discrete auxiliary distribution $\hat{F}$ composed of $n-1$ mass points with mass $\frac{1}{n-1}$ each. Step 2 then concentrates these jumps into a single mass point resulting in F^*. Both steps increase the left hand side of (6).

It is the existing property rights in a service slot which explain the difference between the two outside option mechanisms. The key difference between the random and fcfs initial schedules is that the fcfs order gives players full possession over their time of service (with probability one) while the random order only issues a probabilistic ticket. This concentration of property rights on a single service ticket which comes with the fcfs schedule makes it impossible to efficiently reschedule the queue. The reason is that the agent who is to be served first in the initial schedule knows that he will not 'buy forward' (ie. require earlier service) for sure and thus will not exchange his slot with a marginally higher type behind him for a merely marginal payment.

Discussion

The above results suggest that the insertion of some uncertainty into a deterministic initial queue (thus turning it stochastic) makes efficient rescheduling possible. This idea is further explored in the example of the following subsection. More precisely, consider a lottery which results with probability p in the random queue and probability $(1-p)$ the fcfs queue. Let this lottery be executed if at least one player disagrees in participating in the efficient mechanism.

Corollary 1. *Since the worst-off type in the lottery is continuous in p, for p sufficiently high, there exists an equilibrium in which the efficient allocation is implemented.*

One may wonder whether we can extend our positive result on random queues if we are to consider a more general cost structure than the linear waiting costs we discuss above. This must be answered negatively, as Mitra (2002) shows that it is impossible to simultaneously generalise over linear costs, balancing the budget and require an efficient allocation. Thus our approach is arguably the most general with respect to a cost structure still allowing for a positive result.

Extending the model's assumption of common valuations of the service to private valuations does not change our results, since the players' service valuation affects neither the efficient order nor the incentive compatibility constraints. Finally, it is easy to relax our balanced budget condition to allow for a surplus if that should be desired. The hard constraint is that no outsider should be required to subsidise the mechanism.

Example

To illustrate the above results and the fact that any queue can be efficiently rescheduled if sufficient service uncertainty is injected into the original allocation (corollary 1), we now study the set of efficiently implementable allocations in three player examples based, for simplicity, on the uniform distribution. In order to allow for a graphic interpretation in a simplex diagram, we represent the set of efficiently implementable allocations for exogenously given service probabilities for one player (wlg player 1).

	slot Q1	slot Q2	slot Q3
player P1	p_{11}	p_{12}	$1 - p_{11} - p_{12}$
player P2	p_{21}	p_{22}	$1 - p_{21} - p_{22}$
player P3	$1 - p_{11} - p_{21}$	$1 - p_{12} - p_{22}$	$1 - (1 - p_{11} - p_{12}) - (1 - p_{21} - p_{22})$

We interpret the simplex' corners Q_1, Q_2, Q_3 as the events of being served in slot 1,2,3, respectively, with probability one. For a particular player i, the service probability p_{i1} of being served in the first slot is drawn orthogonal to $\overline{Q_2, Q_3}$ and pointing towards Q_1. Similarly, p_{i2} is the probability of being served in the second slot, and $1 - p_{i1} - p_{i2}$ is the probability of being served last. Fixing the first player, there is a feasible set of probability mass left for players two and three to be served at slot j labelled as q_j, $j = 1, 2, 3$ (the vertical sums in the above table). Since probabilities must sum to one for both players and service slots, it is sufficient to choose one of the remaining players (wlg player 2) and draw the efficiently implementable set of service probabilities—for which (6) is non-positive—for this player.[6]

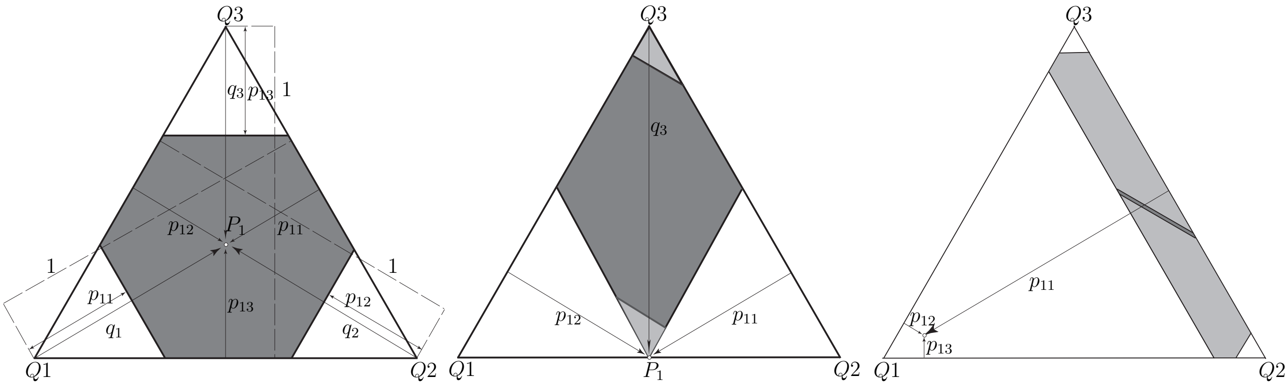

Figure 1: The efficiently implementable region (shaded dark) in terms of service probabilities for player 2 conditional on some service probability P_1 for player 1. Left: $P_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, centre: $P_1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, right: $P_1 = \left(\frac{87.75}{100}, \frac{6.125}{100}, \frac{6.125}{100}\right)$. The set of feasible service probabilities is shaded light.

In figure 1, the feasible region of service probabilities for players 2 and 3 (relative to given service probabilities for player 1 P_1) is drawn light grey. The efficiently implementable region is shaded dark. In accord with proposition 2, the left panel shows that the efficiently implementable set is *identical* to the feasible set for 'random' $P_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. In the centre and right panels the implementable region is a subset of the feasible region. Choosing player 1's service probability p_{11} higher than in the right panel results in the vanishing of the implementable region, a slight reduction of p_{11} widens the implementable region between the two perpendiculars to $\overline{Q_1, Q_3}$.

4 An efficient indirect mechanism

As an illustration of our prior results, we now analyse an auction game that implements the efficient schedule. Our mechanism works on n objects but is otherwise similar to the bidding

[6] This implicitly defines a set of service probabilities for the third player which is not shown in the diagrams.

game studied by Cramton, Gibbons, and Klemperer (1987). We then proceed to argue for individual rationality of the random queue. The auction has the following properties:

1. Each player $i \in \mathcal{N}$ offers some payment for being served in each position of the queue, ie. all players simultaneously offer n-vectors of bids.

2. We assign queue positions $s = 1, \ldots, n$ in increasing order. The highest bidder for position s gets this position and pays the own bid for this slot.[7] The assigned bidder's bids are removed from subsequent slot-allocations.

3. Every slot's payment is shared in equal amounts by *all* other players.

Notice that this simultaneous game is equivalent to a mechanism where the slots are allocated sequentially, starting with the first slot. Every player submits a single bid for the currently auctioned slot as long as the player is still unassigned. No additional information is revealed. In this sequential game, an agent's bid for the k^{th} slot is relevant only if the bidder did not secure service at any previous slot. Thus, given that an agent is still unassigned, he ignores all previous proceedings when deciding on his k^{th} bid. Therefore, denoting by S the set of $n-k$ opponents with the lowest types among the $n-1$ opponents of player i, if the bidders' bidding function is given by the increasing function $\beta^k(\theta_j)$, then agent i submits the bid b for the k^{th} slot which maximises

$$\begin{aligned}
\Pi_i^k(b^k) &= \operatorname{pr}(b > \max_{j\in S}\{\beta^k(\theta_j)\})\,\mathbb{E}\left[-b - k\theta_i + L^W \,\middle|\, b > \max_{j\in S}\{\beta^k(\theta_j)\}\right] + \\
\operatorname{pr}(b &< \max_{j\in S}\{\beta^k(\theta_j)\})\,\mathbb{E}\left[\frac{\max\{\beta^k(\theta_j)\}}{n-1} + L^L \,\middle|\, b < \max_{j\in S}\{\beta^k(\theta_j)\}\right]
\end{aligned}$$

where

$$L^W := \sum_{l>k} \frac{\max\left\{\tilde{\beta}^l(\theta_j)\right\}}{n-1}, \quad L^L := \Pi_i^{k+1}(b^{k+1})$$

and $\max\left\{\tilde{\beta}^l(\theta_j)\right\}$ is the winning bid for slot $l > k$.[8]

Proposition 4. *An equilibrium bidding function of the indirect game described above is increasing in the agent's type and is given by*

$$\beta^k(\theta_i) = \left(\int_0^{\theta_i} (-kx - L^L + L^W)\left(\int_0^x \tilde{F}_k(\theta_j)d\theta_j\right)^{\frac{1}{n-1}} \tilde{F}_k(x)dx\right)\left(\int_0^{\theta_i} \tilde{F}_k(\theta_j)d\theta_j\right)^{-\frac{n}{n-1}} \tag{7}$$

for $k = 1, \ldots, n-1$, *where*

$$\tilde{F}_k(\theta_j) = (F(\theta_j))^{n-k} \sum_{p=0}^{k-1} \binom{n-k+p-1}{n-k-1} (1 - F(\theta_j))^p$$

[7] Ties are broken with equal probability among winners.

[8] Notice that the above $\Pi_i^k(\cdot)$ is not agent i's utility. However, if we want to write agent i's utility as a function only of bids for slot k, we obtain an expression like $A + B\Pi_i^k(\cdot)$, where A and B only depend on the bids for the slots previous to k.

is the distribution of the $n-k$ highest order statistic among $n-1$ variables.

Notice that an agent's payment consists of two parts: He gets the average winning bid of all slots assigned to other players and he pays his own bid for the position at which he is served. Since $\beta^k(\theta_i)$ is an increasing function, this mechanism leads to the efficient allocation.

The effect of some player i slightly increasing his bid for slot k is two-fold: a) the 'standard' effect of increasing the utility through obtaining a higher-valued object in return for a higher payment, and b) the decrease in the sum of transfers from the other players because a player who previously obtained an earlier slot and now gets later service pays less than before.

More precisely, the effects on player i are as follows. If even before the increase, player i obtained slot k, and thus the increased bid leaves the allocation unchanged, player i's payment increases. If the change results in player i obtaining slot k while he would not obtain this slot with the lower bid, then the effects are the following: On the one hand his utility increases through obtaining earlier service while on the other hand the cost is an increased payment and lower transfers from the other bidders. Since these transfers constitute of the individual winning bids of the other players, i's $1/(n-1)$ share of the previously winning bid for slot k is exchanged for his share of a lower payment for a later slot. Thus player i's transfers decrease. It is therefore harder for the indirect mechanism studied above to ensure an increasing bidding function than for a standard auction where winning bids are not redistributed.

Finally, our auction game is also individually rational with respect to the random queue. This is a simple consequence of revenue equivalence: (a) Both the direct and indirect mechanisms balance their budget and hence produce the same revenue. (b) Both mechanisms implement the same efficient allocation. Thus, revenue equivalence tells us that the utility of the lowest type is equal under both mechanisms. Since incentive compatibility implies then that the utility of any type depends only on the utility of the lowest type and the expected service slot, the expected utility of any type under the direct and indirect mechanisms is the same. Therefore the indirect game implements the efficient allocation, balances its budget and is individually rational with respect to the random queue.

Conclusion

We analyse the possibility of rearranging an existing queue into its efficient order through voluntary trade between the queueing agents. Desirable generalisations are over linear waiting costs (eg. multi-dimensional signals) and the equal (unit) processing time assumptions. Well known existing results, however, make us pessimistic about the prospects of such generalisations. In particular, it is known that it is impossible to generalise over linear costs as long as both a balanced budget and efficiency are desired. Another potential generalisation is to extend the model with a stream of stochastically arriving customers and thus turn the scheduling problem into a queueing problem. This will create technical difficulties, but our main conclusion that too strong property rights prevent efficient reordering of the queue will remain in place. Allowing for agents' private information on the time required to complete the service does not make the model more interesting, since this information will be revealed and can be conditioned upon. In case of misrepresenting the service time, fines can be imposed.

Appendix

Proof of lemma 3. In efficient scheduling, the expected waiting time of type θ_i is given by

$$W_i^{ef}(\theta_i, \sigma) = \sum_{k=1}^{n} k \binom{n-1}{k-1} (F(\theta_i))^{n-k} (1 - F(\theta_i))^{k-1} = n + (1-n)F(\theta_i)$$

where the second equality follows from the expectation of the binomial distribution where the success probability of each trial is $1 - F(\theta_i)$, the number of trials is $n-1$, and $k-1$ is the number of successes. $\square$

Proof of theorem 1. Inserting (3) into (5) results in (6). $\square$

Proof of proposition 2. We have to show that, for $\theta^* = \theta^*(\text{ran}) = F^{-1}({}^1\!/_2)$,

$$(1-n)\left[\int_0^{\theta^*} \theta F(\theta) f(\theta) d\theta - \int_{\theta^*}^{1} \theta(1 - F(\theta)) f(\theta) d\theta\right] \geq 0.$$

Integration by parts of the first expression between brackets gives

$$\int_0^{\theta^*} \theta F(\theta) f(\theta) d\theta = \theta (F(\theta))^2 \Big|_0^{\theta^*} - \int_0^{\theta^*} \theta F(\theta) f(\theta) d\theta - \int_0^{\theta^*} (F(\theta))^2 d\theta \tag{8}$$

and integrating the second expression by parts gives

$$\int_{\theta^*}^{1} \theta(1 - F(\theta)) f(\theta) d\theta = -\theta(1 - F(\theta))^2 \Big|_{\theta^*}^{1} - \int_{\theta^*}^{1} \theta(1 - F(\theta)) f(\theta) d\theta + \int_{\theta^*}^{1} (1 - F(\theta))^2 d\theta.$$

Because

$$\theta (F(\theta))^2 \Big|_0^{\theta^*} + \theta(1 - F(\theta))^2 \Big|_{\theta^*}^{1} = 0,$$

we can rewrite the original expression as

$$(1-n)\left[-\int_0^{\theta^*} \frac{(F(\theta))^2}{2} d\theta - \int_{\theta^*}^{1} \frac{(1 - F(\theta))^2}{2} d\theta\right] \geq 0. \qquad \square$$

Proof of proposition 3. Without loss of generality and for notational simplicity, we will assume that player i is served in position i in the initial fcfs schedule. We rewrite (6) as claim of non-implementability as

$$\sum_{i=1}^{n}\left[\int_0^{1} \theta F(\theta) f(\theta) d\theta - \int_{\theta_i^*}^{1} \theta f(\theta) d\theta\right] > 0, \quad \text{for } \theta_i^* = \theta_i^*(\text{fcfs}) = F^{-1}\left(\frac{n-i}{n-1}\right).$$

Using (8) on the first term in brackets and integration by parts on the second term gives

$$-\frac{n}{2}-\frac{n}{2}\int_0^1 (F(\theta))^2 d\theta+\sum_{i=1}^{n}\left[\int_{\theta_i^*}^1 F(\theta)d\theta+\theta_i^*\frac{n-i}{n-1}\right]>0$$

which transforms into

$$\pi\left(F(\theta)\right):=\sum_{i=1}^{n}\int_{\theta_i^*}^1\left[F(\theta)-\frac{n-i}{n-1}\right]d\theta-\frac{n}{2}\int_0^1 (F(\theta))^2 d\theta>0$$

since

$$\int_{\theta_i^*}^1\frac{n-i}{n-1}d\theta=\frac{n-i}{n-1}(1-\theta^*)\quad\text{ and }\quad\sum_{i=1}^{n}\frac{n-i}{n-1}=\frac{n}{2}.$$

For any distribution of types $F(\theta)$, we can thus rewrite (6) as the claim that $\pi\left(F(\theta)\right)>0$. Now define a distribution $F^*(\theta)$, which puts all probability mass at the single point $A\in[0,1]$ and thus removes all uncertainty about the agent's type. Below we show that for any distribution $F(\theta)$ that is different from $F^*(\theta)$, it is true that

$$\pi\left(F(\theta)\right)>\pi\left(F^*(\theta)\right),\quad\text{ where }\quad F^*(\theta)=\begin{cases}0 & \text{if } \theta<A\\ 1 & \text{if } \theta\geq A\end{cases}.\tag{9}$$

Since

$$\pi\left(F^*(\theta)\right)=n\left(1-A\right)-\left(1-A\right)\sum_{i=1}^{n}\frac{n-i}{n-1}-\frac{n}{2}\left(1-A\right)=0\tag{10}$$

for any $A\in[0,1]$, this would complete our proof. We show (9) in two steps. In the first step we show that, for any distribution function $F(\theta)$, it is true that

$$\pi\left(F(\theta)\right)>\pi\left(\hat{F}(\theta)\right)$$

where $\hat{F}(\theta)$ is a distribution function that has no positive measure with positive density and has at most $n-1$ mass points (ie. a discrete distribution). In the second step we show that gathering any two mass points from $\hat{F}(\theta)$ into a single mass point must decrease π.

Step 1. Since in the following we will change the distribution function, denote by $\theta_i^*(F)$ the worst type of player i if the underlying probability is F, which was specified in lemma 2. In this step we show that if, for some i, $\theta_{i+1}^*(F)<\theta_i^*(F)$ then $\pi\left(F(\theta)\right)>\pi\left(\overline{F}(\theta)\right)$ where $\overline{F}(\theta)$ is defined in the following way

$$\overline{F}(\theta)=\begin{cases}F(\theta) & \text{if} \quad \theta<\theta_{i+1}^*(F)\text{ or }\theta\geq\theta_i^*(F)\\ F(\theta_{i+1}^*) & \text{if} \quad \theta_{i+1}^*(F)\leq\theta<b_i\\ F(\theta_i^*) & \text{if} \quad b_i\leq\theta<\theta_i^*(F)\end{cases}$$

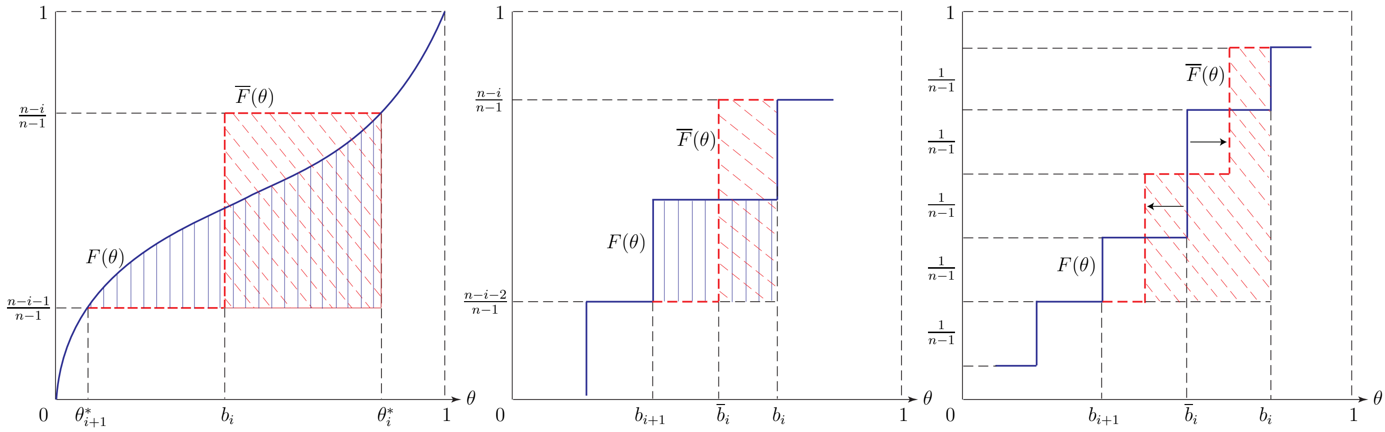

Figure 2: Step 1 (left): The area under the solid $F(\theta)$ is replaced by the equally sized rectangle under the dashed $\overline{F}(\theta)$. Step 2 (centre): Combining two steps of the solid $F(\theta)$ into a single step of equivalent 'virtual' weight. Right: Redistributing a double mass point in $F(\theta)$ into its neighbours.

and b_i is the solution to

$$\frac{1}{n-1}\left(F^{-1}\left(\frac{n-i}{n-1}\right)-b_i\right)=\int_{F^{-1}\left(\frac{n-i-1}{n-1}\right)}^{F^{-1}\left(\frac{n-i}{n-1}\right)}\left(F(\theta)-\frac{n-i-1}{n-1}\right)d\theta.$$

Notice that the boundary points θ^* of the new distribution $\overline{F}(\theta)$ coincide: $\theta_i^*(\overline{F}) = \theta_{i+1}^*(\overline{F}) = b_i$. By choice of b_i the first term of $\pi\left(\overline{F}(\theta)\right)$ does not change, while the change in the second term is

$$\int_0^1 F(\theta)^2 d\theta - \int_0^1 \overline{F}(\theta)^2 d\theta = \int_{F^{-1}\left(\frac{n-i-1}{n-1}\right)}^{F^{-1}\left(\frac{n-i}{n-1}\right)} F(\theta)^2 d\theta -$$
$$\left(\left(b_i - F^{-1}\left(\frac{n-i-1}{n-1}\right)\right)\left(\frac{n-i-1}{n-1}\right)^2 + \left(F^{-1}\left(\frac{n-i}{n-1}\right)-b_i\right)\left(\frac{n-i}{n-1}\right)^2\right).$$

Below we show that $\int_0^1 F(\theta)^2 d\theta - \int_0^1 \overline{F}(\theta)^2 d\theta$ is negative. Notice that the second line of the previous expression can be rewritten as

$$\left(\frac{n-i-1}{n-1}\right)^2\left(F^{-1}\left(\frac{n-i}{n-1}\right)-F^{-1}\left(\frac{n-i-1}{n-1}\right)\right)+\frac{1}{(n-1)^2}\left(F^{-1}\left(\frac{n-i}{n-1}\right)-b_i\right)+$$
$$\frac{2\,(n-i-1)}{(n-1)^2}\left(F^{-1}\left(\frac{n-i}{n-1}\right)-b_i\right)=\left(\frac{n-i-1}{n-1}\right)^2\left(F^{-1}\left(\frac{n-i}{n-1}\right)-F^{-1}\left(\frac{n-i-1}{n-1}\right)\right)+$$
$$\frac{2n-2i-1}{(n-1)}\int_{F^{-1}\left(\frac{n-i-1}{n-1}\right)}^{F^{-1}\left(\frac{n-i}{n-1}\right)}\left(F(\theta)-\frac{n-i-1}{n-1}\right)d\theta.$$

Therefore, we can rewrite $\int_0^1 F(\theta)^2 d\theta - \int_0^1 \overline{F}(\theta)^2 d\theta$ as follows

$$\int_{F^{-1}\left(\frac{n-i-1}{n-1}\right)}^{F^{-1}\left(\frac{n-i}{n-1}\right)} F(\theta)\left(F(\theta) - \frac{2n-2i-1}{(n-1)}\right) d\theta +$$
$$\left(F^{-1}\left(\frac{n-i}{n-1}\right) - F^{-1}\left(\frac{n-i-1}{n-1}\right)\right)\left(\frac{(n-i-1)(2n-2i-1)-(n-i-1)^2}{(n-1)^2}\right) =$$
$$\int_{F^{-1}\left(\frac{n-i-1}{n-1}\right)}^{F^{-1}\left(\frac{n-i}{n-1}\right)} \left[F(\theta)\left(F(\theta) - \frac{2n-2i-1}{(n-1)}\right) + \frac{(n-i-1)(n-i)}{(n-1)^2}\right] d\theta < 0.$$

Where the last inequality follows from the fact that the integrand is zero for the integral limits. Moreover, the integrand is quadratic in $F(\theta)$ and thus has a minimal point.

Our argument allows us to restrict attention to distributions which have at most $n-1$ mass points where the probability of any mass point is $k/(n-1)$ where k is natural number.

Step 2. Note that step 1 allows us to restrict attention to discrete distributions with $n-1$ mass points. After step 1, every mass point has probability $1/(n-1)$.

Since $\pi(F(\theta))$ is continuous in b_i and $1 \geq b_{i-1} \geq b_i \geq b_{i+1} \geq 0$, we can conclude that there exist $b_1^* \geq \ldots \geq b_{n-1}^*$ that minimises $\pi(F(\theta))$. To complete the proof, we show that if there is an i such that $b_i > b_{i+1}$, then $\pi(F(\theta)) > \pi\left(\overline{F}(\theta)\right)$ where $\overline{F}(\theta)$ is defined as

$$\overline{F}(\theta) = \begin{cases} F(\theta) & \text{if } \theta < b_{i+1} \text{ or } \theta \geq b_i \\ \frac{n-i-2}{n-1} & \text{if } b_{i+1} \leq \theta < \overline{b}_i \\ \frac{n-i}{n-1} & \text{if } \overline{b}_i \leq \theta < b_i \end{cases}$$

and $\overline{b}_i$ is given by $\left(\overline{b}_i - b_{i+1}\right)(n-i-1) = \left(b_i - \overline{b}_i\right)(n-i)$ or

$$\overline{b}_i = \frac{b_i(n-i) + b_{i+1}(n-i-1)}{2n-2i-1}.$$

Similarly to the first step, this change does not affect first term of π. Note that

$$\begin{aligned}
&\int_0^1 F(\theta)^2 d\theta - \int_0^1 \overline{F}(\theta)^2 d\theta \\
&= \left(\frac{n-i-1}{n-1}\right)^2 (b_i - b_{i+1}) - \left(\frac{n-i-2}{n-1}\right)^2 \left(\overline{b}_i - b_{i+1}\right) - \left(\frac{n-i}{n-1}\right)^2 \left(b_i - \overline{b}_i\right) \\
&= \left(\frac{n-i-1}{n-1}\right)^2 (b_i - b_{i+1}) - \left(\frac{n-i-1}{n-1}\right)^2 \left(\overline{b}_i - b_{i+1}\right) - \left(\frac{n-i-1}{n-1}\right)^2 \left(b_i - \overline{b}_i\right) \\
&- \frac{1-2(n-i-1)}{(n-1)^2}\left(\overline{b}_i - b_{i+1}\right) - \frac{1+2(n-i-1)}{(n-1)^2}\left(b_i - \overline{b}_i\right) \\
&= -\frac{1}{(n-1)^2}(b_i - b_{i+1}) - \frac{2(n-i-1)}{(n-1)^2}\left(b_i + b_{i+1} - 2\overline{b}_i\right).
\end{aligned}$$

Plugging the definition of $\overline{b}_i$ into the last expression gives us

$$-\frac{1}{(n-1)^2}(b_i - b_{i+1}) + \frac{2(n-i-1)}{(n-1)^2}\frac{b_i - b_{i+1}}{2n-2i-1} = -\frac{b_i - b_{i+1}}{(n-1)^2}\left[1 - \frac{2(n-i-1)}{2n-2i-1}\right] < 0$$

which completes the argument. Notice that after the first application of step 2, the combined mass point has probability mass of $2/(n-1)$. In order to be able to apply step 2 again, one can think of this one point as actually consisting of *two* mass points of equal probability of $1/(n-1)$ each. Reapplying step 2 to combine these into their respective neighbouring mass points then makes no problems. This is illustrated in the right hand panel of figure 2. □

Proof of proposition 4. Agent i chooses b to maximise

$$\begin{aligned} \Pi_i^k(b^k) = \quad & \mathrm{pr}(b > \max_{j\in S}\left\{\beta^k(\theta_j)\right\})\,\mathbb{E}\left[-b - k\theta_i + L^W \middle| \, b > \max_{j\in S}\left\{\beta^k(\theta_j)\right\}\right] + \\ & \mathrm{pr}(b < \max_{j\in S}\left\{\beta^k(\theta_j)\right\})\,\mathbb{E}\left[\frac{\max\left\{\beta^k(\theta_j)\right\}}{n-1} + L^L \middle| \, b < \max_{j\in S}\left\{\beta^k(\theta_j)\right\}\right] \end{aligned} \tag{11}$$

$$\text{where} \quad L^W := \sum_{l>k}\frac{\max\left\{\tilde{\beta}^l(\theta_j)\right\}}{n-1}, \text{ and } L^L := \Pi_i^{k+1}(b^{k+1}).$$

L^W can be interpreted as the slot k winner's utility from the opponents' payments for the slots auctioned after k. L^L is the expected utility a bidder who does not win slot k (or any previous slot) gets from the auctioning of slots after k. Since bidding functions are monotonically increasing, we know that $\mathrm{pr}(b > \max_{j\in S}\left\{\beta^k(\theta_j)\right\}) =$

$$\tilde{F}_k(\beta^{k^{-1}}(b)) := \mathrm{pr}(\theta_j < \beta^{k^{-1}}(b)\forall j \in S) = (F(\beta^{k^{-1}}(b)))^{n-k}\sum_{j=0}^{k-1}\begin{pmatrix} n-k+j-1 \\ n-k-1 \end{pmatrix}\left(1 - F(\beta^{k^{-1}}(b))\right)^j.$$

Using this notation, we can rewrite (11) as

$$\Pi_i^k(b^k) = \int_0^{\beta^{k^{-1}}(b)}(-b - k\theta_i + L^W)\tilde{F}_k(\theta_j)d\theta_j + \int_{\beta^{k^{-1}}(b)}^1\left(\frac{\beta^k(\theta_j)}{n-1} + L^L\right)\tilde{F}_k(\theta_j)d\theta_j.$$

Maximising wrt b gives

$$\begin{aligned} \frac{\partial \Pi_i^k(b^k)}{\partial b} = \quad & -\int_0^{\beta^{k^{-1}}(b)}\tilde{F}_k(\theta_j)d\theta_j + (-b - k\theta_i + L^W)\tilde{F}_k(\beta^{k^{-1}}(b))\frac{1}{\beta^{k\prime}(\hat{\theta})} \\ & -\left(\frac{b}{n-1} + L^L\right)\tilde{F}_k(\beta^{k^{-1}}(b))\frac{1}{\beta^{k\prime}(\hat{\theta})} = 0 \end{aligned}$$

where $\hat{\theta}$ is such that $\beta^k(\hat{\theta}) = b$. This transforms into the ordinary differential equation

$$-\int_0^{\beta^{k^{-1}}(b)} \tilde{F}(\theta_j)d\theta_j - \left(b + k\theta_i + \frac{b}{n-1} + L^L - L^W\right) \frac{\tilde{F}(\beta^{k^{-1}}(b))}{\beta^{k\prime}(\hat{\theta})} = 0.$$

For the initial condition of $\beta(0) = 0$, a solution to this is obtained as

$$\beta^k(\theta_i) = \left(\int_0^{\theta_i} (-kx - L^L + L^W) \left(\int_0^x \tilde{F}(\theta_j)d\theta_j\right)^{\frac{1}{n-1}} \tilde{F}(x)dx\right) \left(\int_0^{\theta_i} \tilde{F}(\theta_j)d\theta_j\right)^{-\frac{n}{n-1}}$$

which equals (7). Checking the slope of this bidding function gives

$$\begin{aligned} \frac{\partial \beta^k(b^k)}{\partial \theta_i} = & \left(-k\theta_i - L^L + L^W\right) \left(\int_0^{\theta_i} \tilde{F}(\theta_j)d\theta_j\right)^{\frac{1}{n-1}} F(\theta_i)d\theta_i \left(\int_0^{\theta_i} \tilde{F}(\theta_j)d\theta_j\right)^{-\frac{n}{n-1}} + \\ & \left(\tilde{F}(\theta_i)\right)^{-\frac{n}{n-1}} \int_0^{\theta_i} (-kx - L^L + L^W) \left(\int_0^x \tilde{F}(\theta_j)d\theta_j\right)^{\frac{1}{n-1}} \tilde{F}(x)dx \end{aligned}$$

where each constituent component is positive since $L^W > L^L + k\theta_i$. □

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