# Simple New Keynesian Model without Capital

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# Outline

- Formulate the nonlinear equilibrium conditions of the model.
  - Need actual nonlinear conditions to study Ramsey-optimal policy, even if we want to use linearization methods to study Ramsey.
    - Ramsey will be used to define 'output gap' in positive model of the economy, in which monetary policy is governed by the Taylor rule.
    - Later, when discussing 'timeless perspective', will discuss use of Ramsey-optimal policy in actual, real-time implementation of monetary policy.
  - Need nonlinear equations if we were to study higher order perturbation solutions.
- Study properties of the NK model with Taylor rule, using Dynare.

# Clarida-Gali-Gertler Model

- Households maximize:

$$E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \; \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}, \; \varepsilon_t^{\tau} \sim iid,$$

- Subject to:

$$P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + T_t$$

- Intratemporal first order condition:

$$C_t \exp(\tau_t) N_t^{\varphi} = \frac{W_t}{P_t}$$

# Household Intertemporal FONC

- Condition:

$$1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \frac{R_t}{1+\pi_{t+1}}$$

– or, for when we do linearize later:

$$1 = \beta E_t \frac{C_t}{C_{t+1}} \frac{R_t}{1+\pi_{t+1}}$$

$$= \beta E_t \exp[\log(R_t) - \log(1+\pi_{t+1}) - \Delta c_{t+1}]$$

$$\simeq \beta \exp[\log(R_t) - E_t \pi_{t+1} - E_t \Delta c_{t+1}],\ c_t \equiv \log(C_t)$$

– take log of both sides:

$$0 = \log(\beta) + r_t - E_t \pi_{t+1} - E_t \Delta c_{t+1},\ r_t = \log(R_t)$$

– or

$$c_t = -\log(\beta) - [r_t - E_t \pi_{t+1}] + c_{t+1}$$

# Final Good Firms

- Buy $Y_{i,t},\ i \in [0,1]$ at prices $P_{i,t}$ and sell $Y_t$ for $P_t$
- Take all prices as given (competitive)
- Profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

- Production function:

$$Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} di,\ \varepsilon > 1,$$

- First order condition:

$$Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \qquad \to P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$

# Intermediate Good Firms

- Each ith good produced by a single monopoly producer.
- Demand curve:

$$Y_{i,t} = Y_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon}$$

- Technology:

$$Y_{i,t} = \exp(a_t) N_{i,t},\ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,$$

- Calvo Price-setting Friction

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t} & \text{with probability } \theta \end{cases},$$

# Marginal Cost

real marginal cost $=$ $$s_t = \frac{\frac{dCost}{dworker}}{\frac{dOutput}{dworker}} = \frac{(1-\nu)W_t/P_t}{\exp(a_t)}$$

$$= \frac{\overbrace{(1-\nu)}^{=\frac{\varepsilon-1}{\varepsilon} \text{ in efficient setting}} \qquad C_t \exp(\tau_t) N_t^{\varphi}}{\exp(a_t)}$$

# The Intermediate Firm's Decisions

- i*th* firm is required to satisfy whatever demand shows up at its posted price.

- It's only real decision is to adjust price whenever the opportunity arises.

# Intermediate Good Firm

- Present discounted value of firm profits:

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\upsilon_{t+j}}^{\text{marginal value of dividends to household}=u_{c,t+j}/P_{t+j}} \overbrace{\left[ \overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} s_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}^{\text{period } t+j \text{ profits sent to household}}$$

- Each of the $1-\theta$ firms that can optimize price choose $\tilde{P}_t$ to optimize

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\theta^j}^{\substack{\text{in selecting price, firm only cares about} \\ \text{future states in which it can't reoptimize}}} \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}].$$

# Intermediate Good Firm Problem

- Substitute out the demand curve:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon}].$$

- Differentiate with respect to $\tilde{P}_t$ :

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [(1-\varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1}] = 0,$$

- or

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

# Intermediate Good Firm Problem

- Objective:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t},\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j} \bar{\pi}_{t+j-1} \cdots \bar{\pi}_{t+1}},\ j \geq 1 \\ 1,\ j = 0. \end{cases},\ X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}},\ j > 0$$

# Intermediate Good Firm Problem

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0$$

- Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

- But, still need expressions for $K_t$, $F_t$.

$$
\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left( \frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}}^{\text{exactly } K_{t+1}!} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}
\end{aligned}
$$

- From previous slide:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

- Substituting out for marginal cost:

$$\frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{W_t/P_t}^{d\text{Cost}/d\text{labor}}}{\underbrace{\overbrace{\exp(a_t)}}_{}^{d\text{Output}/d\text{labor}}}$$

$$= \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \frac{\overbrace{\exp(\tau_t) N_t^{\varphi} C_t}^{= \frac{W_t}{P_t} \text{ by household optimization}}}{\exp(a_t)}.$$

# In Sum

- solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- Where:

$$K_t = (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^{\varphi} C_t}{\exp(a_t)} + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1}$$

# To Characterize Equilibrium

- Have equations characterizing optimization by firms and households.
- Still need:
  - Expression for all the prices. Prices, $P_{i,t},\ 0 \leq i \leq 1$ , will all be different because of the price setting frictions.
  - Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t} N_t, \text{ in general}$$

# Going for Prices

- Aggregate price relationship

Calvo insight:
This is just a simple function of last period's aggregate price because non-optimizers chosen at random.

$$P_t = \left[\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

$$= \left[\int_{\text{firms that reoptimize price}} P_{i,t}^{(1-\varepsilon)} di + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

$$\overbrace{=}^{\text{all reoptimizers choose same price}} \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di\right]^{\frac{1}{1-\varepsilon}}$$

- In principle, to solve the model need all the prices, $P_t,\ P_{i,t},\ 0 \leq i \leq 1$
  - Fortunately, that won't be necessary.

# Expression for $\tilde{p}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[ (1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

  – Only two variables here!

- Divide by $P_t$:

$$1 = \left[ (1-\theta)\tilde{p}_t^{(1-\varepsilon)} + \theta \left( \frac{1}{\bar{\pi}_t} \right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t = \left[ \frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}$$

# Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no 'aggregate production function' in this model
  - If you know how many people are working, *N*, and the state of technology, *a*, you don't have enough information to know what *Y* is.
  - Price frictions imply that resources will not be efficiently allocated among different inputs.
    - Implies *Y* low for given *a* and *N*. How low?
    - Tak Yun (JME) gave a simple answer.

# Tak Yun Algebra

$$Y_t^* = \int_0^1 Y_{i,t} di \left( = \int_0^1 A_t N_{i,t} di \overbrace{=}^{\text{labor market clearing}} A_t N_t \right)$$

$$\overbrace{=}^{\text{demand curve}} Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \int_0^1 (P_{i,t})^{-\varepsilon} di$$

Calvo insight

$$= Y_t P_t^{\varepsilon} (P_t^*)^{-\varepsilon}$$

- Where:

$$P_t^* \equiv \left[ \int_0^1 P_{i,t}^{-\varepsilon} di \right]^{\frac{-1}{\varepsilon}} = [(1-\theta)\tilde{P}_t^{-\varepsilon} + \theta (P_{t-1}^*)^{-\varepsilon}]^{\frac{-1}{\varepsilon}}$$

# Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

$$Y_t = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} Y_t^*$$

$$= p_t^* e^{a_t} N_t,$$

- 'efficiency distortion':

$$p_t^* : \begin{cases} \leq 1 \\ = 1 \quad P_{i,t} = P_{j,t}, \text{ all } i,j \end{cases}$$

# Collecting Equilibrium Conditions

- Price setting:

$$K_t = (1-\nu)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t)N_t^{\varphi}C_t}{A_t} + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon}K_{t+1} \quad (1)$$

$$F_t = 1 + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon-1}F_{t+1} \quad (2)$$

- Intermediate good firm optimality and restriction across prices:

$$\overbrace{\frac{K_t}{F_t}}^{=\tilde{p}_t \text{ by firm optimality}} = \overbrace{\left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \quad (3)$$

# Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

$$p_t^* = \left[ (1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4)$$

- Household Intertemporal Condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

- Aggregate inputs and output:

$$C_t = p_t^* e^{a_t} N_t \quad (6)$$

- 6 equations, 8 unknowns:

$$v,\ C_t,\ p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$$

- System under determined!

# Underdetermined System

- Not surprising: we added a variable, the nominal rate of interest.
- Also, we’re counting subsidy as among the unknowns.
- Have two extra policy variables.
- One way to pin them down: compute optimal policy.

# Ramsey-Optimal Policy

- 6 equations in 8 unknowns…..
  - Many configurations of the 8 unknowns that satisfy the 6 equations.
  - Look for the best configurations (Ramsey optimal)
    - Value of tax subsidy and of *R* represent optimal policy

- Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.

# Ramsey Problem

$$\max_{\nu, p_t^*, C_t, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{1t} \left[ \frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]$$

$$+ \lambda_{4t} \left[ (1-\nu) \frac{\varepsilon}{\varepsilon-1} \frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[ F_t \left( \frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right]$$

$$+ \lambda_{6t} [C_t - p_t^* e^{a_t} N_t] \}$$

# Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{\nu, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

defines $R$ →

$$+ \lambda_{1t} \left[ \frac{1}{p_t^* N_t} - E_t \frac{e^{a_t} \beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

defines $F$ →

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]$$

defines tax →

$$+ \lambda_{4t} \left[ (1-\nu) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

defines $K$ →

$$+ \lambda_{5t} \left[ F_t \left( \frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \}$$

# Solving the Ramsey Problem, cnt'd

- Simplified problem:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1-\theta) \left( \frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \}$$

- First order conditions with respect to $p_t^*,\ \bar{\pi}_t,\ N_t$

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^{\varepsilon} = \lambda_{2t},\ \bar{\pi}_t = \left[ \frac{(p_{t-1}^*)^{\varepsilon-1}}{1-\theta+\theta(p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}},\ N_t = \exp\left( -\frac{\tau_t}{\varphi+1} \right)$$

- Substituting the solution for inflation into law of motion for price distortion:

$$p_t^* = \left[ (1-\theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}.$$

# Solution to Ramsey Problem

Eventually, price distortions eliminated, regardless of shocks

$$p_t^* = \left[(1-\theta) + \theta(p_{t-1}^*)^{(\varepsilon-1)}\right]^{\frac{1}{(\varepsilon-1)}}$$

When price distortions gone, so is inflation.

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

Efficient ('first best') allocations in real economy

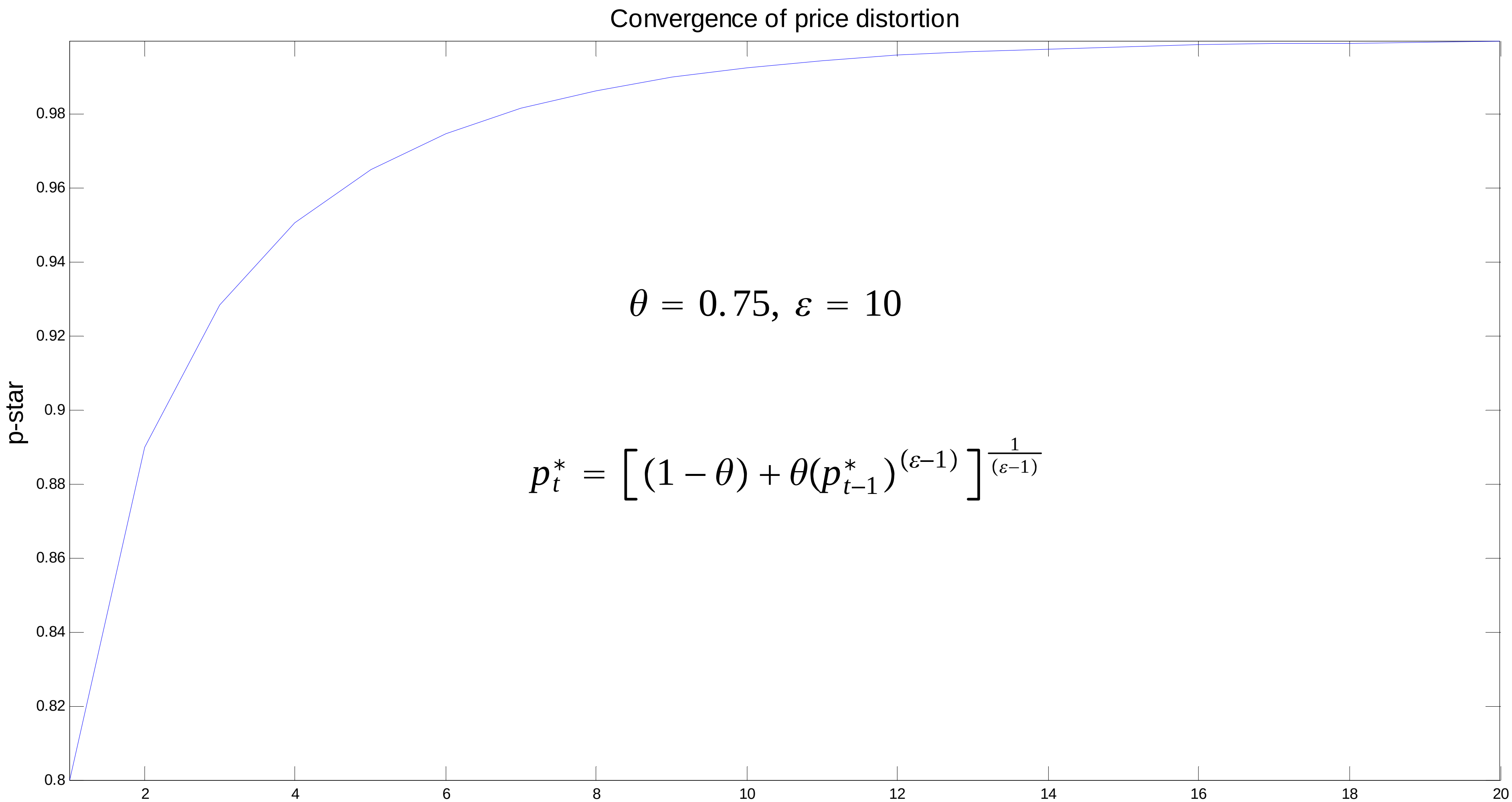
$$N_t = \exp\left(-\frac{\tau_t}{1+\varphi}\right)$$

$$1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$

Consumption corresponds to efficient allocations in real economy, eventually when price distortions gone

# Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide



- The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)

- Refer to the Ramsey allocations as the ‘natural allocations’….
  - Natural consumption, natural rate of interest, etc.

# Equations of the NK Model Under the Optimal Policy (‘Natural Equilibrium’)

- Output and employment is (eventually)

$$y_t^* = a_t - \frac{1}{1+\varphi}\tau_t,\ n_t^* = -\frac{1}{1+\varphi}\tau_t$$

- Intertemporal Euler equation after taking logs and ignoring variance adjustment term:

$$y_t^* = -[r_t^* - E_t\pi_{t+1}^* - rr] + E_t y_{t+1}^*,\ rr = -\log\beta$$

- Inflation in Ramsey equilibrium is (eventually) zero.

# Solving for Natural Rate of Interest

- Intertemporal euler equation in natural equilibrium:

$$\overbrace{a_t - \frac{1}{1+\varphi}\tau_t}^{y_t^*} = -[r_t^* - rr] + E_t \overbrace{\left(a_{t+1} - \frac{1}{1+\varphi}\tau_{t+1}\right)}^{y_{t+1}^*}$$

- Back out the natural rate:

$$r_t^* = rr + \rho\Delta a_t + \frac{1}{1+\varphi}(1-\lambda)\tau_t$$

- Shocks:

$$\tau_t = \lambda\tau_{t-1} + \varepsilon_t^\tau,\ \Delta a_t = \rho\Delta a_{t-1} + \varepsilon_t$$

# Next, Put Turn to the NK Model with Taylor Rule

# Taylor Rule

- Taylor rule: designed, so that in steady state, inflation is zero ( $\bar{\pi} = 1$ )

- Employment subsidy extinguishes monopoly power in steady state:

$$(1 - \nu)\frac{\varepsilon}{\varepsilon - 1} = 1$$

# NK *IS* Curve

- Euler equation in two equilibria:

$$\text{Taylor rule equilibrium: } y_t = -[r_t - E_t\pi_{t+1} - rr] + E_t y_{t+1}$$

$$\text{Natural equilibrium: } y_t^* = -[r_t^* - rr] + E_t y_{t+1}^*$$

- Subtract:

Output gap

$$x_t = -[r_t - E_t\pi_{t+1} - r_t^*] + E_t x_{t+1}$$

# Output in NK Equilibrium

- Agg output relation:

$$y_t = \log p_t^* + n_t + a_t, \ \log p_t^* = \begin{cases} = 0 & \text{if } P_{i,t} = P_{j,t} \text{ for all } i,j \\ \leq 0 & \text{otherwise} \end{cases} .$$

- To first order approximation,

$$\hat{p}_t^* \approx \theta \hat{p}_{t-1}^* + 0 \times \bar{\pi}_t, \ (\rightarrow p_t^* \approx 1)$$

# Price Setting Equations

- Log-linearly expand the price setting equations about steady state.

$$1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta\theta F_{t+1} - F_t = 0 \qquad F_t \left( \frac{1-\theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t = 0$$

$$(1-\nu)\frac{\varepsilon}{\varepsilon-1}\frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta\theta\bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t = 0$$

- Log-linearly expand about steady state:

$$\hat{\bar{\pi}}_t = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\varphi)x_t + \beta\hat{\bar{\pi}}_{t+1}$$

- See http://faculty.wcas.northwestern.edu/~lchrist/course/solving_handout.pdf

# Taylor Rule

- Policy rule

$$r_t = \alpha r_{t-1} + (1-\alpha)[rr + \phi_\pi \pi_t + \phi_x x_t] + u_t, \; , \; x_t \equiv y_t - y_t^*$$

# Equations of Actual Equilibrium Closed by Adding Policy Rule

$$\beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \text{ (Phillips curve)}$$

$$-[r_t - E_t \pi_{t+1} - r_t^*] + E_t x_{t+1} - x_t = 0 \text{ (IS equation)}$$

$$\alpha r_{t-1} + (1-\alpha)\phi_\pi \pi_t + (1-\alpha)\phi_x x_t - r_t = 0 \text{ (policy rule)}$$

$$r_t^* - \rho \Delta a_t - \frac{1}{1+\varphi}(1-\lambda)\tau_t = 0 \text{ (definition of natural rate)}$$

# Solving the Model

$$s_t = \begin{pmatrix} \Delta a_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_t^{\tau} \end{pmatrix}$$

$$s_t = P s_{t-1} + \epsilon_t$$

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ r_{t+1}^* \end{pmatrix} + \begin{bmatrix} -1 & \kappa & 0 & 0 \\ 0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\ (1-\alpha)\phi_{\pi} & (1-\alpha)\phi_x & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr_t^* \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ x_{t-1} \\ r_{t-1} \\ r_{t-1}^* \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s_{t+1} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\sigma\psi\rho & -\frac{1}{\sigma+\varphi}(1-\lambda) \end{pmatrix} s_t$$

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

# Solving the Model

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

- Solution:

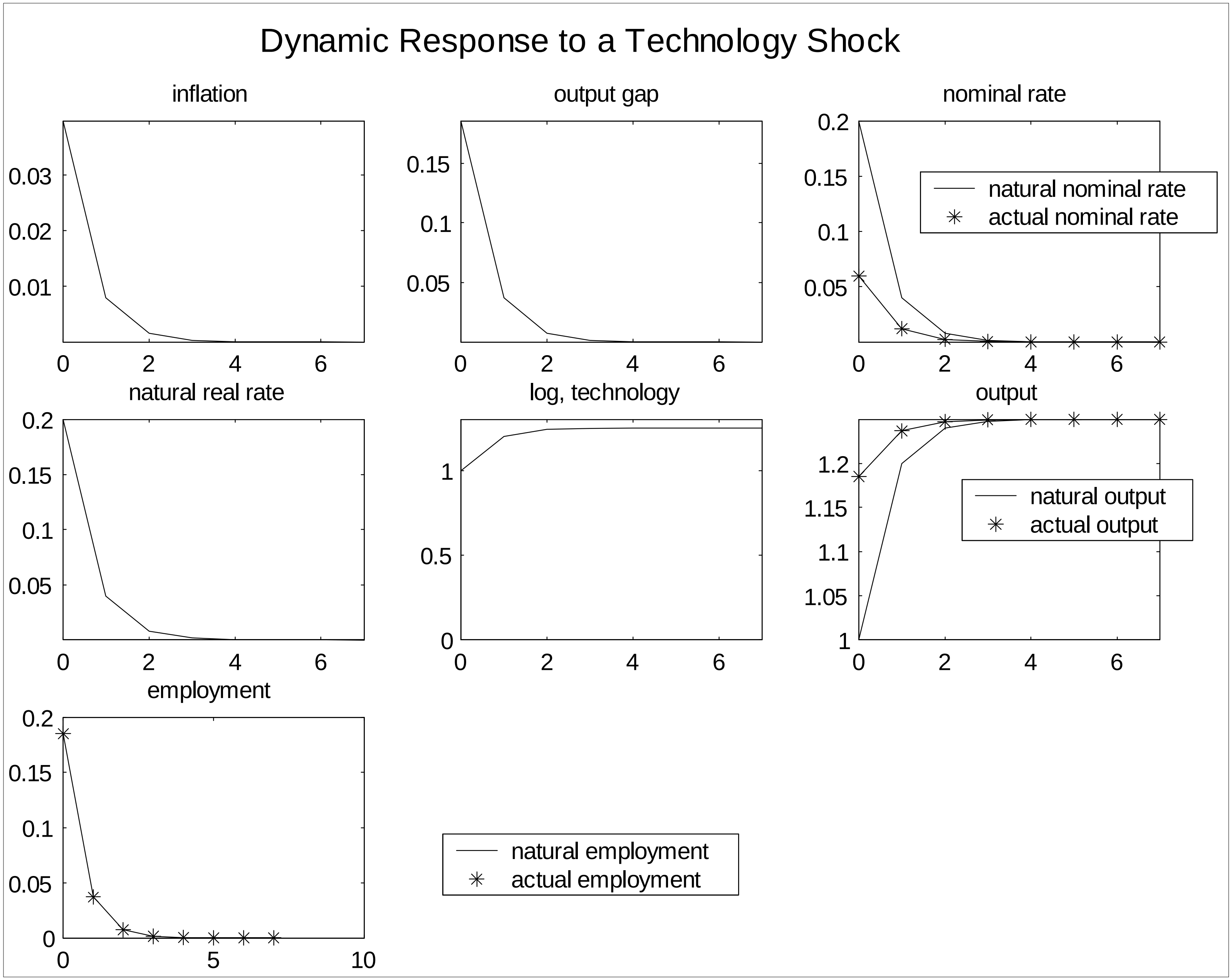
$$z_t = A z_{t-1} + B s_t$$

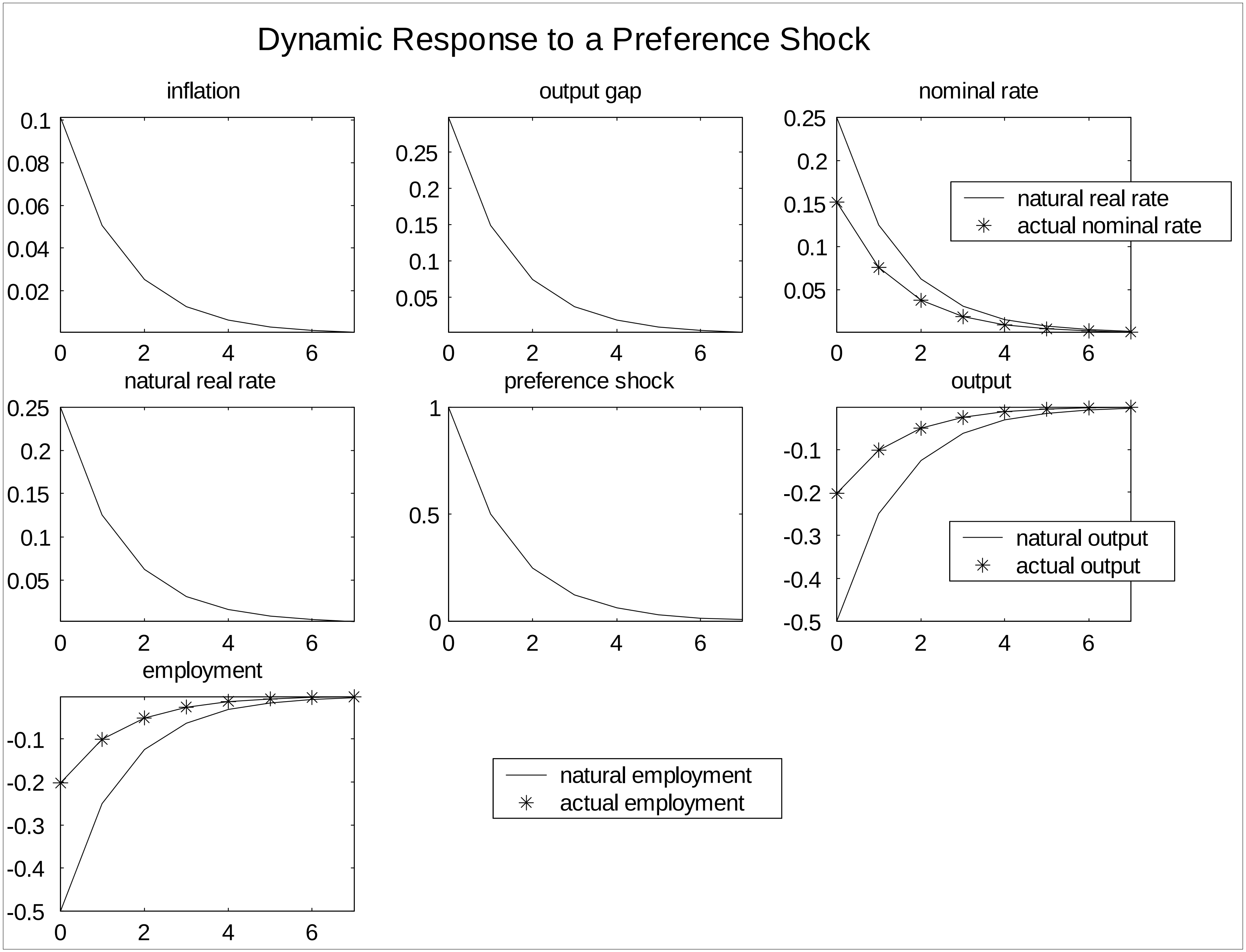
- As before:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0,$$

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

$\phi_x = 0,\ \phi_\pi = 1.5,\ \beta = 0.99,\ \varphi = 1,\ \rho = 0.2,\ \theta = 0.75,\ \alpha = 0,\ \delta = 0.2,\ \lambda = 0.5.$




Dynamic Response to a Preference Shock
inflation
output gap
nominal rate
natural real rate
actual nominal rate
natural real rate
preference shock
output
natural output
actual output
employment
natural employment
actual employment

# Conclusion of NK Model Analysis

- We studied examples in which the Taylor rule moves the interest rate in the right direction in response to shocks.

- However, the move is not strong enough. Will consider modifications of the Taylor rule using Dynare.