Incentive Compatible Allocation and Exchange of Discrete Resources*

Marek Pycia† M. Utku Ünver‡

UCLA *Boston College*

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Abstract

We study the allocation and exchange of discrete resources without monetary transfers. In market design literature, some problems that fall in this category are the house allocation problem with and without existing tenants, and the kidney exchange problem. We introduce a new class of direct mechanisms that we call "trading cycles with brokers and owners," and show that (i) each mechanism in the class is group strategy-proof and Pareto efficient, and (ii) each group strategy-proof and Pareto-efficient direct mechanism is in the class. As a corollary we show that in pure exchange problems the well-known class of top trading cycles mechanisms contains all group strategy-proof, efficient, and individually rational mechanisms.

Keywords: Mechanism design, group strategy-proofness, Pareto efficiency, matching, house allocation, house exchange, outside options.

JEL classification: C78, D78

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†UCLA, Department of Economics, 8283 Bunche Hall, Los Angeles, CA 90095. E-mail: pycia@econ.ucla.edu

‡Boston College, Department of Economics, 140 Commonwealth Ave., Chestnut Hill, MA 02467. E-mail: unver@bc.edu

1 Introduction

The theory and practical applications involving the allocation and exchange of indivisible resources without monetary transfers have recently been attracting attention of economists. Market designers have tailored new models and mechanisms to solve real-life problems such as the allocation of students to on-campus dormitory rooms at US colleges [cf. Abdulkadiroğlu and Sönmez, 2003] and exchanges of live donor kidney transplants [cf. Roth et al., 2004].

There are common features of these real-life problems. There is a group of agents each of whom would like to consume a single indivisible object to which we will refer to as a house using the terminology coined by Shapley and Scarf [1974]. Agents have strict preferences over the houses. Some of the houses are agents' common endowment, while others belong to private endowments of the agents. The outcome of the problem is a matching of agents and houses. Since we provide a unified treatment of both house allocation (from social endowment) and house exchange (among agents with private endowments), we refer to our environment as house allocation and exchange. We study direct revelation mechanisms, that is, agents reveal their preferences over houses, and the mechanism matches each agent with a house (or agent's outside option).

The direct mechanisms studied in the literature have two essential properties: group strategy-proofness and Pareto efficiency.[1] Group strategy-proofness means that no group of agents can jointly manipulate so that all of them weakly benefit from this manipulation, while at least one in the group strictly benefits. Such mechanisms are not only non-manipulable but also impose minimal computational costs on the participants and do not discriminate agents based on their ability to strategize and their access to information [cf. Vickrey, 1961, Dasgupta et al., 1979, Pathak and Sönmez, 2008].

We introduce a new class of direct mechanisms that we call trading cycles with brokers and owners (or simply trading cycles), and show that (i) each mechanism in the class is group strategy-proof and Pareto efficient, and (ii) each group strategy-proof and Pareto-efficient direct mechanism can be implemented through a mechanism from the class. Thus, we characterize the full class of relevant direct mechanisms, and lay down the structure of the house allocation and exchange problem. The new trading-cycles-with-brokers-and-owners mechanisms can be used to address design problems that were beyond the reach of the previously known mechanisms.

[1]The group strategy-proofness is the right strategy-proofness concept in applications such as kidney exchange, see Section 6. In our environment, group strategy-proofness may be formulated as a non-cooperative property: it is equivalent to dominant-strategy incentive compatibility and non-bossiness. For a group strategy-proof mechanism, Pareto efficiency is equivalent to assuming that each allocation is feasible, and thus is implied by, for instance, unanimity. See Section 2.2 for details.

A trading-cycles-with-brokers-and-owners algorithm matches houses and agents in a sequence of rounds. At each round some agents and houses are matched and removed from the problem. At the beginning of the round, each previously unmatched house is controlled by an unmatched agent. We distinguish two forms of control over a house which we call ownership and brokerage (at any round, there is at most one broker and one brokered house). Each house points to the agent that controls it, and each agent points to his most preferred unmatched house. The only exception is the broker (if there is one) who points to his most preferred unmatched house other than the brokered house. In the resultant directed graph, there exists at least one exchange cycle. Each agent in each exchange cycle is matched with the house he points to.[2]

The allocation of control rights in each round is fully determined by how agents and houses were matched prior to that round. The above-described procedure takes as given the mapping from partial matchings to control rights. Each such mapping that satisfies certain consistency conditions determines a mechanism in our class. For expositional purposes, we first formulate our main result for settings in which all houses are social endowments, and hence there are no additional exogenous constraints on the allocation of control rights [cf. Hylland and Zeckhauser, 1979]. For instance, at some universities, the dormitory rooms are treated as social endowments. At other universities however, some students, such as sophomores, have the right to stay in the room they lived in the preceding year. In kidney exchange, patients (interpreted as agents) come with a paired-donor (interpreted as a house) and have to be matched with at least their paired-donor. Such exogenous control rights are straightforwardly accommodated by our mechanism class, and we derive corollaries of our main result for problems in which some houses are private endowments of agents and the participation in the mechanism has to be individually rational. The class of group strategy-proof, efficient, and individually rational direct mechanism equals the class of individually rational trading cycles mechanisms. Trading cycle mechanisms are individually rational if and only if they may be represented by a consistent control right structure in which each agent is given ownership rights over all houses from his endowment.

The above class of mechanisms is built on the top-trading cycles idea attributed to David Gale by Shapley and Scarf [1974], and developed by Abdulkadiroğlu and Sönmez [1999] and Pápai [2000]. The subclass of our mechanisms without brokers was introduced by Pápai [2000]; it is the largest class of group strategy-proof and Pareto-efficient mechanisms previously known. As a corollary of our main results, we give an elegant characterization of top trading cycles in environments in which

[2]We study environments without outside options as well as environments with outside options. The results are the same in both environments but the above algorithm needs to be slightly generalized in the case of outside options by allowing agents to point to houses or their outside options. We also need to postpone matching a broker with his outside option till a round an agent who owns a house lists the brokered house as his most preferred one.

each agent has a nonempty endowment, a condition satisfied for instance in kidney exchange of Roth et al. [2004]. In such environments top trading cycles mechanisms are the unique mechanisms which are group strategy-proof, efficient, and individually rational.

On the technical side, our main innovations lies in introducing the brokerage control rights. Previously only ownership control rights were studied in the context of house allocation and exchange. Recognizing the role of brokers in house allocation and exchange is crucial to obtaining the entire class of group strategy-proof and Pareto-efficient mechanisms. The introduction of brokers is also useful in some design problems.

As an example of a mechanism design problem in which brokerage rights are useful, consider a manager who assigns n tasks $t_1, ..., t_n$ to n employees $w_1, ..., w_n$ with strict preferences over the tasks. The manager wants the allocation to be Pareto efficient with regard to the employees' preferences. Within this constraint, she would like to avoid assigning task t_1 to employee w_1. She wants to use a group strategy-proof direct mechanism, because she does not know employees' preferences. The only way to do it using the previously known mechanisms is to endow employees $w_2, ..., w_n$ with the tasks, let them find the Pareto-efficient allocation through a top-trading cycles procedure, such as hierarchical exchange of Pápai [2000], and then allocate the remaining task to employee w_1. Ex ante each such procedure is unfair to the employee w_1. Using a trading-cycles-with-brokers-and-owners mechanism, the manager can achieve her objective without the extreme discrimination of the employee w_1. To do so, she makes w_1 the broker of t_1, allocates the remaining tasks among $w_2, ..., w_n$ (for instance she may make w_i the owner of t_i, $i = 2, ..., n$), and runs trading cycles with brokers and owners. The allocation of employee w_1 in this trading-cycles-with-brokers-and-owners mechanism is better than in any top-trading cycles procedures satisfying manager's constraints; the allocation is weakly better regardless of agents' preference profile, and it is strictly better for some preference profiles.

There are many studies that characterize desirable properties of house allocation and exchange through variants of top-trading cycles mechanisms. The most general class of mechanisms in the literature prior to our study was constructed by Pápai [2000]. Her class characterizes group strategy-proofness and Pareto efficiency together with an additional property which she refers to as reallocation-proofness. A mechanism is reallocation-proof in the sense of Papai (2000) if there does not exist a profile of preferences, a pair of agents and a pair of preference misrepresentations such that (i) if both of them misrepresent their preferences, both of them weakly gain and one of them strictly gain by swapping their assignments, and (ii) if only one of them misrepresents his preferences, he cannot change his assignment. Papai also notes that the stronger reallocation-proofness-type property obtained by dropping condition (ii) conflicts with group strategy-proofness and Pareto efficiency. We do not use reallocation-proofness in our results.[3]

In matching and house allocation and exchange literature, the standard modeling approach has been to use strict preferences instead of the full preference domain. Participants are frequently allowed to submit only strict preference orderings to real-life direct mechanisms in various markets, such as dormitory room allocation, school choice, matching of interns and hospitals. As Ehlers [2002] shows "one cannot go much beyond strict preferences if one insists on efficiency and group strategy-proofness." He characterizes group strategy-proof and Pareto-efficient mechanisms in the maximal subset of full preference domain such that such a mechanism exists. The full preference domain gives rise to an impossibility result, i.e., when agents can be indifferent among houses, there exists no mechanism that is group strategy-proof and Pareto efficient. Under strict preferences, his class of mechanisms is a subclass of ours, and substantially different from the general class.[4]

The study of strategy-proof mechanisms has a long tradition. Gibbard [1973] and Satterthwaite [1975] have shown that all strategy-proof and unanimous voting rules are dictatorial. Satterthwaite and Sonnenschein [1981] extended this result to public good economies with production, and Zhou [1991] extended it to pure public good economies. In social choice models, Dasgupta et al. [1979] have proved that every Pareto-efficient and strategy-proof social choice rule is dictatorial. In exchange economies, Barberà and Jackson [1995] showed that strategy-proof mechanisms are Pareto-inefficient.

Even with additional structure, it has been difficult to characterize Pareto-efficient and strategy-proof mechanisms that are non-dictatorial. Such characterizations have been obtained by Green and Laffont [1977] for decision problems with monetary transfers and quasi-linear utilities [cf. Vickrey, 1961, Clarke, 1971, Groves, 1973, Roberts, 1979], Barberà et al. [1997] for sharing a perfectly divisible good among agents with single-peaked preferences over their shares, and by Barberà et al. [1993] for voting problems with single-peaked preferences [cf. Moulin, 1980].[5]

[3]Ma [1994], Svensson [1999], Ergin [2000], Miyagawa [2002], Ehlers et al. [2002], Ehlers and Klaus [2003, 2007], Kesten [2009], Velez-Cardona [2008], Sönmez and Ünver [2010] characterize subclasses of group strategy-proof, Pareto-efficient, and reallocation-proof mechanisms.

[4]See Bogomolnaia et al. [2005] for another characterization with indifferences.

[5]Sönmez [1999] studies generalized matching problems in which each agent is endowed with a good. The class of such problems non-trivially intersects with the class of house allocation and exchange problems studied in this paper. He shows that there exists a Pareto-efficient, strategy-proof, and individually rational mechanism if and only if the core is nonempty and agents are indifferent between all core allocations. He also shows that any such mechanism is group strategy-proof [cf. Shapley and Scarf, 1974, Roth and Postlewaite, 1977, Roth, 1982, Ma, 1994].

2 Model of House Allocation (The Case of Social Endowments)

2.1 Environment

Let I be a set of **agents** and H be a set of **houses.** We use letters i, j,k to refer to agents and h, g,e to refer to houses. Each agent i has a **strict preference relation** over H, denoted by $\succ_i$.[6] Let $\mathbf{P}_i$ be the set of strict preference relations for agent i, and let $\mathbf{P}_J$ denote the Cartesian product $\times_{i\in J}\mathbf{P}_i$ for any $J \subseteq I$. Any profile from $\succ = (\succ_i)_{i\in I}$ from $\mathbf{P} \equiv \mathbf{P}_I$ is called a **preference profile**. For all $\succ \in \mathbf{P}$ and all $J \subseteq I$, let $\succ_J = (\succ_i)_{i\in J} \in P_J$ be the restriction of $\succ$ to J.

To simplify the exposition, we initially make two assumptions. Both of these assumptions are fully relaxed in subsequent sections. First, we initially restrict attention to house allocation problems. A **house allocation problem** is the triple $\langle I, H, \succ\rangle$. Throughout the paper, we fix I and H, and thus, a problem is identified with its preference profile. In Section 6, we generalize the setting and the results to house allocation and exchange by allowing agents to have initial rights over houses. The results on allocation and exchange turn out to be straightforward corollaries of the results on (pure) allocation. Second, we initially follow the tradition adopted by many papers in the literature [cf. Svensson, 1999, Bogomolnaia and Moulin, 2001] and assume that $|H| \geq |I|$ so that each agent is allocated a house. This assumption is satisfied in settings in which each agent is always allocated a house (there are no outside options), as well as in settings in which agents' outside options are tradeable, effectively being indistinguishable from houses. In Section 7, we allow for non-tradeable outside options and show that analogues of our results remain true irrespective of whether $|H| \geq |I|$ or $|H| < |I|$.

An outcome of a house allocation problem is a matching. To define a matching, let us start with a more general concept that we will use frequently. A **submatching** is an allocation of a subset of houses to a subset of agents, such that no two different agents get the same house. Formally, a submatching is a one-to-one function $\sigma : J \to H$; where for $J \subseteq I$, using the standard function notation, we denote by $\sigma(i)$ the assignment of agent $i \in J$ under σ, and by $\sigma^{-1}(h)$ the agent that got house $h \in \sigma(J)$ under σ. Let $\mathcal{S}$ be the set of submatchings. For each $\sigma \in \mathcal{S}$, let I_σ denote the set of agents matched by σ and $H_\sigma \subseteq H$ denote the set of houses matched by σ. For all $h \in H$, let $\mathcal{S}_{-h} \subset \mathcal{S}$ be the set of submatchings $\sigma \in \mathcal{S}$ such that $h \in H - H_\sigma$, i.e., the set of submatchings at which house h is unmatched. In virtue of the set-theoretic interpretation of functions, submatchings are sets of agent-house pairs, and are ordered by inclusion. A **matching** is a maximal submatching, that is $\mu \in \mathcal{S}$ is a matching if $I_\mu = I$. Let $\mathcal{M} \subset \mathcal{S}$ be the set of matchings. We will write $\overline{I_\sigma}$ for $I - I_\sigma$, and $\overline{H_\sigma}$ for $H - H_\sigma$ for short. We will also write $\overline{\mathcal{M}}$ for $\mathcal{S} - \mathcal{M}$.

[6]By $\succeq_i$ we denote the induced weak preference relation; that is, for any $g, h \in H$, $g \succeq_i h \Longleftrightarrow g = h$ or $g \succ_i h$.

A **(direct) mechanism** is a mapping $\varphi : \mathbf{P} \longrightarrow \mathcal{M}$ that assigns a matching for each preference profile (or, equivalently, allocation problem).

2.2 Group Strategy-Proofness and Pareto Efficiency

A mechanism is group strategy-proof if there is no group of agents that can misstate their preferences in a way such that each one in the group gets a weakly better house, and at least one agent in the group gets a strictly better house. Formally, a mechanism φ is **group strategy-proof** if for all $\succ \in \mathbf{P}$, there exists no $J \subseteq I$ and $\succ'_J \in \mathbf{P}_J$ such that

$$\varphi[\succ'_J, \succ_{-J}](i) \ \succeq_i \ \varphi[\succ](i) \text{ for all } i \in J,$$

and

$$\varphi[\succ'_J, \succ_{-J}](j) \succ_j \varphi[\succ](j) \text{ for at least one } j \in J.$$

In our domain group strategy-proofness has a non-cooperative interpretation, and is equivalent to the conjunction of two non-cooperative properties: individual strategy-proofness and non-bossiness. Strategy-proofness of a mechanism means that the truthful revelation of preferences is a weakly dominant strategy: a mechanism φ is (**individually**) **strategy-proof** if for all $\succ \in \mathbf{P}$, there is no $i \in I$ and $\succ'_i \in \mathbf{P}_i$ such that

$$\varphi[\succ'_i, \succ_{-i}](i) \ \succ_i \ \varphi[\succ](i).$$

Non-bossiness [Satterthwaite and Sonnenschein, 1981] means that no agent can misreport his preferences in such a way that his allocation is not changed but the allocation of some other agent is changed: a mechanism φ is **non-bossy** if for all $\succ \in \mathbf{P}$, there is no $i \in I$ and $\succ'_i \in \mathbf{P}_i$ such that

$$\varphi[\succ'_i, \succ_{-i}](i) = \varphi[\succ](i) \quad \text{and} \quad \varphi[\succ'_i, \succ_{-i}] \neq \varphi[\succ].$$

The following lemma due to Pápai [2000] states the non-cooperative interpretation of group strategy-proofness:

Lemma 1. *Pápai [2000]A house-allocation mechanism is group strategy-proof if and only if it is individually strategy-proof and non-bossy.*

Another useful formulation of group strategy-proofness builds on Maskin [1999]. A mechanism φ is **Maskin-monotonic** if $\varphi[\succ'] = \varphi[\succ]$ whenever $\succ' \in \mathbf{P}$ is a φ-monotonic transformation of $\succ \in \mathbf{P}$. A preference profile $\succ' \in \mathbf{P}$ is a **φ-monotonic transformation** of $\succ \in \mathbf{P}$ if

$$\{h \in H : h \succeq_i \varphi[\succ](i)\} \supseteq \{h \in H : h \succeq'_i \varphi[\succ](i)\} \text{ for all } i \in I.$$

Thus, for each agent, the set of houses better than the base-profile allocation weakly shrinks when we go from the base profile to its monotonic transformation. The following lemma is due to Takamiya [2001]:

Lemma 2. *[Takamiya, 2001] A house-allocation mechanism is Maskin-monotonic if and only if it is group strategy-proof.*

A matching is Pareto efficient if no other matching would make everybody weakly better off, and at least one agent strictly better off. That is, a matching $\mu \in \mathcal{M}$ is Pareto efficient if there exists no matching $\nu \in \mathcal{M}$ such that for all $i \in I$, $\nu(i) \succeq_i \mu(i)$, and for some $i \in I$, $\nu(i) \succ_i \mu(i)$. A mechanism is **Pareto efficient** if it finds a Pareto-efficient matching for every problem.

Pareto efficiency is a very weak requirement when imposed on group strategy-proof mechanisms. Every group strategy-proof mechanism that maps $\mathbf{P}$ onto the entire set of matchings $\mathcal{M}$ is Pareto efficient. This surjectivity property is implied for instance by unanimity of Gibbard [1973] and Satterthwaite [1975]. A house allocation mechanism is **unanimous** if the mechanism allocates all agents their most-preferred houses whenever no two agents put the same house as their most-preferred choice (that is the overall matching most preferred by all agents obtains whenever the agents agree on what is the most preferred matching).

3 Beyond Top Trading Cycles

3.1 Top Trading Cycles

To set the stage for our trading-cycles-with-brokers-and-owners (TCBO) mechanism, let us look at the well-known top trading cycles (TTC) algorithm adapted by Pápai [2000] to house allocation problems.[7] The class of mechanisms presented in this section is identical to Pápai's "hierarchical exchange" class. Our presentation however is novel and aims to simultaneously simplify the earlier constructions of Pápai's class, and to introduce some of the terminology we will later use to introduce our class of all group strategy-proof and efficient mechanisms (TCBO).

TTC is a recursive algorithm that matches houses to agents in a sequence of rounds. In each round some agents and houses are matched. The matches will not be changed in subsequent rounds of the algorithm.

At the beginning of each round, each unmatched house is "owned" by an unmatched agent. The algorithm creates a directed graph in which each unmatched house points to the agent who owns it,

[7] The algorithm was originally proposed by David Gale for the special case of house exchange [cf. Shapley and Scarf, 1974].

and each unmatched agent points to his most preferred house among the unmatched houses. In the resultant directed graph there exists at least one exchange cycle in which agent 1's most preferred house is owned by agent 2, agent 2's most preferred house is owned by agent 3, ..., and finally, for some $k = 1, 2, ...$, agent k's most preferred house is owned by agent 1. Moreover, no two exchange cycles intersect. The algorithm matches all agents in exchange cycles with their most preferred houses.

The algorithm terminates when all agents are matched. As at least one agent-house pair is matched in every round, the algorithm terminates after finitely many rounds.

As we see the outcome of the TTC algorithm is determined by two types of inputs: agents' preferences and agents' rights of ownership over houses. The preferences are, of course, submitted by the agents. The ownership rights are defined exogenously as part of the mechanism.[8] We formalize this aspect of the mechanism via the following concept.

Definition 1. A **structure of ownership rights** is a collection of mappings $\left\{c_\sigma : \overline{H_\sigma} \to \overline{I_\sigma}\right\}_{\sigma \in \overline{\mathcal{M}}}$. The structure of ownership rights $\{c_\sigma\}_{\sigma \in \overline{\mathcal{M}}}$ is **consistent** if

$$c_\sigma^{-1}(i) \subseteq c_{\sigma'}^{-1}(i) \text{ if } \sigma \subseteq \sigma' \in \overline{\mathcal{M}} \text{ and } i \in \overline{I_{\sigma'}}.$$

The structure of ownership rights tells us at each submatching which unmatched agent owns any particular unmatched house. Agent i owns house h at submatching σ when $c_\sigma(h) = i$. Consistency means that whenever an agent owns a house at a submatching (σ) then he also owns it at any larger submatching (σ') as long as he is unmatched.

Each consistent structure of ownership rights $\{c_\sigma\}_{\sigma \in \overline{\mathcal{M}}}$ determines a *hierarchical exchange mechanism* of Pápai [2000]. This class of mechanisms consists of mappings from agents' preferences $\mathbf{P}$ to matchings $\mathcal{M}$ obtained by running the TTC algorithm with consistent structures of ownership rights. Because of this, we will also refer to hierarchical exchange as **TTC mechanisms**. Pápai showed that all TTC mechanisms are group strategy-proof and Pareto efficient.

As an example consider the TTC mechanism to allocate four houses $h_1, ..., h_4$ to three agents $i_1, ..., i_3$ given by the structure of ownership rights that allocates ownership of houses according to the following table:

h_1	h_2	h_3	h_4
i_1	i_2	i_3	i_1
i_3	i_1	i_2	i_3
i_2	i_3	i_1	i_2

[8] Recall that we are studying an allocation problem in which objects are a collective endowment. In Section 6 we will enlarge the analysis to include exchange problems among agents with private endowments. In exchange problems, some of the mechanism's ownership rights are determined by individual rationality constraints.

That is: for example, house h_1 is initially owned by i_1; at submatchings i_1 is matched, it and i_3 are not matched, it is owned by i_3; at submatchings i_1 and i_3 are matched and it is not matched, it is owned by i_2. Notice that the owner is uniquely determined and that the ownership structure is consistent.

To see how the TTC algorithms run, let us apply this mechanism to the preference profile in which all agents i have the same preferences $\succ_i$:

$$\text{agent } i \text{ preferences: } h_1 \succ_i h_2 \succ_i h_3 \succ_i h_4.$$

In the first round, all agents point to house h_1, houses h_1 and h_4 point to agent i_1, house h_2 points to i_2, and house h_3 points to i_3. In this round, there is one exchange cycle, in which i_1 is matched with h_1.

In the second round, agents i_2,i_3 and houses h_2, h_3, h_4 are unmatched. House h_2 is still owned by i_2 while houses h_3, h_4 are still owned by i_3. In the resultant directed graph, there is again one exchange cycle in which i_2 points to h_2 and h_2 points to i_2, and they are matched.

In the third round agent i_3 owns all unmatched houses, is matched with h_3, and the algorithm terminates.

The second round of this example illustrates two phenomena. First, we cannot allocate the ownership unconditionally as this would leave unresolved the ownership of house h_4 after its initial owner, agent i_1, is matched with house h_1. Second, it illustrates the need for the consistency condition. If the ownership structure was not consistent, and say h_2 was owned by i_3 at $\sigma = \{(i_1, h_1)\}$ (that is after i_1 left with h_1), then agent i_2 would like to misreport his preferences and claim that he prefers h_2 over all other houses.

While under the above preference structure, all exchange cycles involve only one agent and one house, this is not generally true. Consider, for instance, the following preference profile in which i_2 preference between h_2 and h_3 is reversed,

$$\text{agent } i_1 \text{ preferences: } h_1 \succ_{i_1} h_2 \succ_{i_1} h_3 \succ_{i_1} h_4,$$
$$\text{agent } i_2 \text{ preferences: } h_1 \succ_{i_2} h_3 \succ_{i_2} h_2 \succ_{i_3} h_4,$$
$$\text{agent } i_3 \text{ preferences: } h_1 \succ_{i_3} h_2 \succ_{i_3} h_3 \succ_{i_3} h_4.$$

When this profile is reported, the first round is the same as above, but the exchange cycle in the second round has agent i_2 pointing to h_3, h_3 pointing to i_3, and i_3 pointing to h_2.

To appreciate the generality of the Pápai's class, notice that the serial dictatorship of Satterthwaite and Sonnenschein [1981] and Svensson [1994] is a special case of the TTC mechanisms in which at each submatching there is an agent who owns all unmatched houses.

3.2 Beyond Top Trading Cycles: An Example

How a group strategy-proof and efficient non-TTC mechanism might look like? To give an example, we will modify the TTC mechanism for three agents $i_1, ..., i_3$ and three houses $h_1, ..., h_3$ and an ownership structure that allocates ownership of houses according to the following table (obtained by dropping house h_4 in the ownership structure of the example of Subsection 3.2):

h_1	h_2	h_3
i_1	i_2	i_3
i_3	i_1	i_2
i_2	i_3	i_1

The owner is uniquely determined and the ownership structure is consistent. Given this structure let us run TTC with one modification: agent i_1 is not allowed to point to house h_1 as long as there are other unmatched agents. In rounds with other unmatched agents (and hence other unmatched houses), agent i_1 will point to his most preferred house among unmatched houses other than h_1.[9]

For instance, if each agent i has the preference $h_1 \succ_i h_2 \succ_i h_3$ then in the first round agents i_2 and i_3 will point to h_1 but agent i_1 will point to his second choice house, h_2. We will then have an exchange cycle, in which i_1 is matched with h_2 and i_2 is matched with h_1. In the second round, the algorithm matches agent i_3 and house h_3, and terminates.

This mechanism is group strategy-proof and Pareto efficient. An easy recursion may convince us that at each round the submatching formed is Pareto efficient for matched agents. Indeed, if an agent matched in the first round does not get his top choice then he gets the second choice and getting the first choice would hurt another agent matched in that round. In general, agents matched in the n-th round get their first or second choice among houses available in the n-th round, and giving one of these agents a better house would hurt some other agent matched at the same or earlier round. The intuition behind its group strategy-proofness is more complex and we skip discussing it until our formal results.

This mechanism turns out to be different from all TTC mechanisms. To see this point, first observe that the mechanism matches house h_1 with agent i_2 under the illustrative preference profile analyzed above, while it would match h_1 with another agent, i_3, if agent i_1 submitted preferences $h_1 \succ_{i_1} h_3 \succ_{i_1} h_2$ (and other agents $i \neq i_1$ continued to have preferences $h_1 \succ_i h_2 \succ_i h_3$). However, any TTC mechanism would match h_1 with the same agent in these two preference profiles. Indeed,

[9] Pápai [2000] gives an example of a non-TTC mapping from $\mathbf{P}$ to $\mathcal{M}$. Her construction is different from ours though the resultant mappings are identical. As we will show in the next section, the advantage of our construction lies in its generalizability to cover the whole class of group strategy-proof and efficient mechanisms.

TTC ownership structure uniquely determines the agent who owns h_1 at the empty submatching, and this agent would be matched with h_1 in the first round of the algorithm under any preference profile in which all agents put h_1 as their first choice.

For future use, let us notice that in the above example, agent i_1 does not have the full ownership right over h_1. Unless he is the only agent left, he cannot form the trivial exchange cycle that would match him with h_1. He does have some control right over h_1 however: he can trade h_1 for houses owned by other agents. In our general trading-cycles-with-brokers-and-owners algorithm, we will refer to such weak control rights as "brokerage".

4 Trading Cycles with Brokers and Owners

We are now turning to our new algorithm, trading cycles with brokers and owners (TCBO), an example of which we have seen in the previous section. Like TTC, the TCBO is a recursive algorithm that matches agents and houses in exchange cycles over a sequence of rounds. TCBO is more flexible however as it allows two types of intra-round control rights over houses that agents bring to the exchange cycles: ownership and brokerage.

In our description of the TTC class, each TTC mechanism was determined by a consistent ownership structure. Similarly, each TCBO mechanism is determined by a consistent structure of control rights.

Definition 2. A **structure of control rights** is a collection of mappings

$$\left\{(c_\sigma, b_\sigma) : \overline{H_\sigma} \to \overline{I_\sigma} \times \{\text{ownership,brokerage}\}\right\}_{\sigma\in\overline{\mathcal{M}}}.$$

The functions c_σ of the control rights structure tell us which unmatched agent controls any particular unmatched house at submatching σ. Agent i **controls** house $h \in \overline{H_\sigma}$ at submatching σ when $c_\sigma(h) = i$. The type of control is determined by functions b_σ. We say that the agent $c_\sigma(h)$ **owns** h at σ if $b_\sigma(h)$ =ownership, and that the agent $c_\sigma(h)$ **brokers** h at σ if $b_\sigma(h)$ =brokerage. In the former case we call the agent an **owner** and the controlled house an **owned house**. In the latter case we use terms **broker** and **brokered house**. Notice that each controlled (owned or brokered) house is unmatched at σ, any unmatched house is controlled by some uniquely determined agent.

The consistency requirement on TCBO control rights structures consists of three constraints on brokerage at any given submatching (the *within-round* requirements) and three constraints on how the control rights are related across different submatchings (the *across-rounds* requirements).

Within-round Requirements. Consider any $\sigma \in \overline{\mathcal{M}}$.

(**R1**) There is at most one brokered house at σ.

(**R2**) If i is the only unmatched agent at σ then i owns all unmatched houses at σ.

(**R3**) If agent i brokers a house at σ, then i does not own any houses at σ.

The conditions allow for different houses to be brokered at different submatchings, even though there is at most one brokered house at any given submatching.

Requirements R1-R2 are what we need for the TCBO algorithm to be well defined (R3 is necessary for Pareto efficiency and individual strategy-proofness, see Appendix A). With these requirements in place, we are ready to describe the TCBO algorithm, postponing the introduction of the remaining consistency requirements till the next section.

The TCBO algorithm. The algorithm consists of a finite sequence of rounds $r = 1, 2, ...$ In each round some agents are matched with houses. By σ^{r-1} we denote the submatching of agents and houses matched before round r. Before the first round the submatching is empty that is $\sigma^0 = \varnothing$. If $\sigma^{r-1} \in \mathcal{M}$ that is when every agent is matched with a house, the algorithm terminates and gives matching σ^{r-1} as its outcome. If $\sigma^{r-1} \in \overline{\mathcal{M}}$ then the algorithm proceeds with the following three steps of *round* r:

Step 1. Pointing. Each house $h \in \overline{H_{\sigma^{r-1}}}$ points to the agent who controls it at σ^{r-1}. If there exists a broker at σ^{r-1}, then he points to his most preferred house among the ones owned at σ^{r-1}. Every other agent $i \in \overline{I_{\sigma^{r-1}}}$ points to his most preferred house in $\overline{H_{\sigma^{r-1}}}$.

Step 2. Trading cycles. There exists $n \in \{1, 2, ...\}$ and an exchange cycle

$$h^1 \to i^1 \to h^2 \to ...h^n \to i^n \to h^1$$

in which agents $i^\ell \in \overline{I_{\sigma^{r-1}}}$ point to houses $h^{\ell+1} \in \overline{H_{\sigma^{r-1}}}$ and houses h^ℓ points to agents i^ℓ (here $\ell = 1, ..., n$ and superscripts are added modulo n);

Step 3. Matching. Each agent in each trading cycle is matched with the house he is pointing to; σ^r is defined as the union of σ^{r-1} and the set of newly matched agent-house pairs.

The algorithm terminates when all agents or all houses are matched.

Looking back at the example of the previous section we see that it was TCBO and that agent i_1 brokered house h_1 while other agents owned houses. We may now also see that requirements R1 and R2 are needed to make sure that in Step 1 there always is an owned house for the broker to point to. The difference between TTC and TCBO is encapsulated in Step 1, the other steps are standard and were present already in Gale's TTC idea [Shapley and Scarf, 1974]. The existence of the trading cycle follows from there being a finite number of nodes (agents and houses), each pointing at another. The matching of Step 3 is well defined as (i) each agent points to exactly one house, and (ii) each matched house is allocated to exactly one agent (no two different agents pointing to the same house h can belong to trading cycles because there is a unique pointing path that starts with house h. Finally, since we match at least one agent-house pair in every round, and since there are finitely many agents and houses, the algorithm stops after finitely many rounds.

Our algorithm builds upon Gale's top-trading-cycles idea described in Section 3.1, but allows more general trading cycles than top cycles. In TCBO, brokers do not necessarily point to their top choice houses. In contrast, all previous developments of the Gale's idea such as the top-trading cycles algorithm with newcomers [Abdulkadiroğlu and Sönmez, 1999], hierarchical exchange [Pápai, 2000], top trading cycles for school choice [Abdulkadiroğlu and Sönmez, 2003], and top trading cycles and chains algorithm [Roth et al., 2004] allowed only top trading cycles and had all agents point to their top choice among unmatched houses. All these previous developments may be viewed as using a subclass of TCBO in which all control rights are ownership rights and there are no brokers.[10]

The terminology of owners and brokers is motivated by a trading analogy. In each round of the algorithm, an owner can either be matched with the house he controls or with another house obtained from an exchange. A broker cannot be matched with the house he controls; the broker can only be matched with a house obtained from an exchange with other agents. At any submatching (but not globally throughout the algorithm), we can think of the broker of house h as representing a latent agent who owns h but prefers any other house over it. The analogy is of course imperfect and, ultimately, our choice of terminology is arbitrary.

5 Main Results on Allocation

Introduced in the previous section, the TCBO algorithm with a control right structure satisfying R1-R3 gives a Pareto-efficient mechanism mapping profiles from $\mathbf{P}$ to matchings in $\mathcal{M}$. The recursive

[10]In particular, TCBO can easily handle private endowments as explained in Section 6.

argument for efficiency of the non-TTC mechanism from Section 3.2 applies.

Proposition 1. *The TCBO algorithm produces a Pareto-efficient matching for all control rights structures that satisfy R1-R3.*

We are about to see that the TCBO-induced mapping is group strategy-proof if the control right structure also satisfies the following across-round consistency requirements.

> **Across-round Requirements.** Consider any submatchings σ, σ' such that $|\sigma'| = |\sigma| + 1$ and $\sigma \subset \sigma' \in \overline{\mathcal{M}}$, and any agent $i \in \overline{I_{\sigma'}}$
>
> (**R4**) If i owns house h at σ then i owns h at σ'.
>
> (**R5**) Assume that at least two agents from $\overline{I_{\sigma'}}$ own houses at σ. If i brokers house h at σ then i brokers h at σ'.
>
> (**R6**) Assume that there is exactly one agent $i' \in \overline{I_{\sigma'}}$ who owns a house at σ. If i brokers house h at σ but does not broker it at σ' then i' owns h at σ', and i owns all houses in $c_{\sigma}^{-1}(i')$ (that is all houses i' owned at σ) at $\sigma' \cup \{(i', h)\}$.

The requirements R4 and R5 postulate that control rights persist: agents hold on to control rights as we move from smaller to larger submatchings, or through the rounds of the algorithm. R4 (*persistency of ownership*) is identical to the consistency assumption we imposed on TTC. The first example of Section 3.1 illustrated why we need such persistency assumption for the resultant mechanism to be individually strategy-proof. A similar example might convince us that individual strategy-proofness relies also on requirement R5 (*persistency of brokerage*), see appendix. The requirement R6 includes a stipulation that whenever an agent i' takes over the control of the brokered house, then the broker i is in line for ownership rights of i' after i' is matched (*i becomes the heir to i'*). For this reason we sometime call this requirement a *broker-to-heir transition*. R6 is needed to guarantee both non-bossiness and individual strategy-proofness of the mechanisms, see Appendix A.

To sidestep the complications of R5 and R6 in the first reading, the reader is invited to keep in mind a smaller class of control right structures in which both of these requirements are replaced by the following strong form of brokerage persistence: "If $|\sigma'| < |I| - 1$ and agent i brokers house h at σ then i brokers h at σ'." We think that by restricting attention to this smaller class of control right structures, we are not missing much of the flexibility of the TCBO class of mechanism. We must stress however that the complication is there for a reason: there are group strategy-proof and efficient mechanisms that cannot be replicated by TCBO control right structures satisfying the above strengthening of R5-R6 (the relevant example is presented in Appendix A).

We are now ready to define our mechanism class and state our main results.

Definition 3. A control right structure is **consistent** if it satisfies requirements R1-R6. The class of **TCBO mechanisms** (trading cycles with brokers and owners) consists of mappings from agents' preference profiles $\mathbf{P}$ to matchings $\mathcal{M}$ obtained by running the TCBO algorithm with consistent control right structures.

The TTC mechanisms of Section 3.1 and the non-TTC mechanism of Section 3.2 are examples of TCBO. We will denote by $\psi^{c,b}$ the TCBO mechanism obtained from a consistent control right structure $(c_\sigma, b_\sigma)_{\sigma \in \overline{\mathcal{M}}}$.

We will now show that the TCBO class of mechanisms coincides with the class of Pareto-efficient and group strategy-proof direct mechanisms.

Theorem 1. *Every TCBO mechanism is group strategy-proof and Pareto efficient.*

Theorem 2. *Every group strategy-proof and Pareto-efficient direct mechanism is TCBO.*

Before discussing the proof of Theorem 1, let us make the following observation about the TCBO algorithm.

Lemma 3. *If an agent i is unmatched at a round r of the algorithm under preference profiles $[\succ_i, \succ_{-i}]$ and $[\succ'_i, \succ_{-i}]$, then the control rights structure is the same under $[\succ_i, \succ_{-i}]$ and $[\succ'_i, \succ_{-i}]$.*

The lemma obtains because under the assumption of the lemma it must be that $\sigma^{r-1}[\succ_i, \succ_{-i}] = \sigma^{r-1}[\succ'_i, \succ_{-i}]$ that is the same submatching was formed before round r. Hence, also the control rights structures must be the same at round r.

The lemma has an important implication: as long as an agent is unmatched, he cannot influence when he becomes an owner, a broker, or enters the broker-to-heir transition (see R6) by choosing what preferences to submit.

Beginning of the proof of Theorem 1. Proposition 1 demonstrates Pareto efficiency. By Lemma 1, to prove group strategy-proofness it is enough to show that every TCBO mechanism is individually strategy-proof and non-bossy. We will prove individual strategy proofness below, and non-bossiness in Appendix B. Let $\psi^{c,b}$ be a TCBO mechanism. Let $\succ$ be a preference profile. We fix an agent

$i \in I$. We will show that i cannot benefit by submitting $\succ_i' \neq \succ_i$ while the other agents submit $\succ_{-i}$. Let s be the round i leaves (with house h) at $\succ_i$ and s' be the time i leaves (with h') at $\succ_i'$ in the algorithm. We will consider two cases.

Case 1. $s \leq s'$**:** At round s, same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by a straightforward inductive application of Lemma 3. If i is not a broker at time s under $\succ_i$, then, by submitting $\succ_i$, agent i gets the top house among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

Assume now that i is a broker at time s under $\succ_i$. Let e be the brokered house at time s. If e is not agent i's top choice house remaining under $\succ_i$, then by submitting $\succ_i$, agent i gets the top house among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

It remains to consider the situation in which e is broker i's top choice remaining house, and to show that i cannot get e by submitting the profile $\succ_i'$. For an argument through contradiction, assume that under $\succ_i'$ agent i leaves at round s' with house e. Because agent i is a broker when he leaves at $\succ_i$, there is an agent j who is matched with house e at time s. At this time, j is an owner of some owned house h_j, and e is his top choice house. By Lemma 3, the control rights structure at round s is the same under both $\succ_i$ and $\succ_i'$. Hence, i is also a broker at time s after submitting $\succ_i'$, and j is an owner of h_j. Moreover, j's top choice is still house e. That means that under $\succ_i'$ agent j will stay unmatched till $s'+1$. Since agent i leaves with e at s', he cannot be the broker of e at this round, because a broker cannot leave with the brokered house, while another owner j is unmatched. Thus, there is a round $s'' \in \{s+1, ..., s'\}$ at which agent i stops being the broker of e. Since e is still unmatched at this round, there is a broker-to-heir transition between $s''-1$ and s'' (by R6). Because j is an owner of h_j at both $s''-1$ and s'', he would have inherited e at s'' (by R6). Then, however, j would have left with e at s'', as e is j's top choice among houses left at s (and hence those left at s''). A contradiction.

Case 2. $s > s'$**:** At round s', same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by Lemma 3. Consider round s' at both $\succ_i$ and $\succ_i'$. Under $\succ_i'$, agent i points to house $h' = h^1$ that points to agent i^1 that points to ... that points to object h^n that points to agent $i = i^n$ (and this cycle leaves at round s'). If the cycle is trivial ($n = 1$) and h' points back to i, then i owns h'. Since ownership persist by R4, i will own h' at $s > s'$, and thus at round s, agent i would leave with a house at least as good as h'.

In the sequel, assume that there is at least one other agent i^n in the cycle (that is $n \geq 2$).

If each house h^ℓ is owned by i^ℓ, for all $\ell \in \{1, ..., n\}$, then the chain $h' = h^1 \to i^1 \to h^2 \to ... \to h^n \to i$ will stay in the system as long as i is in the system (by persistency of ownership implied through R4). Thus, at round s agent i would leave with a house at least as good as h' under $\succ_i$.

If i^ℓ brokers h^ℓ for some $\ell \in \{1, ..., n\}$, then the chain $h' = h^1 \to i^1 \to h^2 \to ... \to h^n \to i$ will stay in the system as long as i^ℓ continues brokering h^ℓ (since there are no other brokerages and the ownerships persist by R4). If i^ℓ brokers h^ℓ at round s under $\succ_i$, then we are done, since the same cycle would have formed. Thus suppose that at a round $s'' \in \{s'+1, ..., s\}$ broker i^ℓ loses his broker status. Because $n \geq 2$, agent $i^{\ell+1}$ is an owner both at rounds $s''-1$ and s''. Hence, the loss of brokerage status means that i^ℓ enters broker-to-heir transition. We must then have $n = 2$ (since by R6, only 1 previous owner can remain unmatched during broker-to-heir transition). There are two cases: either i^1 owns $h^1 = h'$ and h^2 (and $i^2 = i^\ell$ is the heir) or $i^2 = i$ owns h^1 and h^2. In the former case, i^1 who wants h^2, will leave with it at round s'' under $\succ_i$, and i will inherit $h^1 = h'$ at $s''+1$ by R6. In the latter case, i owns $h^1 = h'$ already at round s''. In both cases, at $s \geq s''$ agent i can only leave with a house at least as good as h' under $\succ_i$. **QED**

Let us finish this section with an overview of the proof of Theorem 2 (the proof is in Appendix C). In the proof we fix a coalitionally strategy-proof and Pareto-efficient direct mechanism φ and construct a TCBO mechanism $\psi^{c,b}$ that is equivalent to φ. We proceed in three steps: we first construct the candidate control rights structure (c, b), then show it satisfies conditions R1-R6, and finally show that the resultant TCBO mechanism $\psi^{c,b}$ equals φ.

We define a candidate control right structure in terms of how φ allocates objects for preferences from some special preference classes. To see how this is done consider an empty submatching and a house h. If φ were a TCBO and h was owned by an agent then at all preference profiles in which all agents rank h as their most preferred house, φ would allocate h to the same agent – the owner of h at the empty submatching. We thus check whether φ allocates h to the same agent at all above profiles, and if it does, we call this agent the candidate owner of h (in the proof, for shortness, we refer to candidate owner as owner*). If φ does not allocate h to the same agent at all above profiles, h is a candidate brokered house. Notice, that if φ were a TCBO and h was brokered by an agent then at every profile at which every agent ranks h as his most preferred house and some other house h' as his second most preferred house, φ would allocate h' to the same agent – the broker of h at the empty submatching. We thus check whether there is an agent who always gets his second most preferred house at the above profiles, and if there is such an agent we call this agent the candidate broker of h (broker* for shortness). Finally, we prove the key lemma that every house h either has a candidate owner or a candidate broker.

The construction of candidate control rights at non-empty other submatchings is similar. The only modification is that instead of looking at preferences at which all agents agree on their most preferred house (or two most preferred houses), we impose this commonality only on unmatched agents, and at the same time assume the matched agents rank the houses they are matched with at the top, while all other agents rank matched houses at the bottom. Thanks to the simplifying assumption that $|H| \geq |I|$, Pareto efficiency of TCBO mechanisms implies that the above procedure would work well if φ was a TCBO, and we prove that indeed it works well whenever φ is group strategy-proof and efficient.[11]

The second step of the proof is to show that the above candidate control right structure indeed satisfies properties R1-R6. The argument is non-trivial and we flesh it out in several lemmas. These lemmas show that the candidate control right structure is indeed consistent. Thus, we have constructed a TCBO mechanism $\psi^{c,b}$. The last step of the proof is to show that $\psi^{c,b} = \varphi$. We rely on the recursive structure of TCBO, and proceed by induction with respect to the rounds of $\psi^{c,b}$.

6 Allocation and Exchange (The Case of Private and Social Endowments)

In this section, we generalize the model by allowing agents to have private endowments. The characterizations in the resultant allocation and exchange domains are straightforward corollaries of our main results. We also relate the results to allocation and exchange market design environments.

6.1 Model of House Allocation and Exchange

Let $\mathcal{H} = \{H_i\}_{i \in \{0\} \cup I}$ be a collection of $|I| + 1$ pairwise-disjoint subsets of H (some of which might be empty) such that $\cup_{i \in \{0\} \cup I} H_i = H$. We interpret houses from H_0 as social endowment of the agents, and houses from H_i, $i \in I$, as private endowment of agent i. A **house allocation and exchange problem** is a list $\langle H, I, \mathcal{H}, \succ \rangle$. Since we allow some of the agents to have empty endowment, the allocation model of Section 2 is contained as a special case with $\mathcal{H} = \{H, \emptyset, ..., \emptyset\}$. We may fixed H, I and $\mathcal{H}$, and identify the house allocation and exchange problem just by its preference profile $\succ$. Matchings and mechanisms are defined as in the allocation model of Section 2.

Pareto efficiency and group strategy-proofness are defined in the same way as in Section 2. In particular, the equivalence between group strategy-proofness and the conjunction of individual strategy-proofness and non-bossiness continues to hold true. In addition to efficiency and strategy-proofness, satisfactory mechanisms in this problem domain should be individually rational. A mechanism is **individually rational** if it always selects an individually rational matching. A matching is individually rational, if it assigns each agent a house that is at least as good as the house he would choose

[11] This point in the construction requires more care in the case $|H| < |I|$, see Appendix **??**.

from among his endowment. Formally a matching μ is individually rational if

$$\mu(i) \succeq_i h \qquad \forall i \in I, \forall h \in H_i.$$

For agents with empty endowments, $H_i = \emptyset$, this condition is tautologically true.

6.2 Results

Our main characterization result for house allocation and exchange is now an immediate corollary of Theorems 1-2.

Theorem 3. *In house allocation and exchange problems, a mechanism is* individually rational*,* Pareto efficient, *and* group strategy-proof *if and only if it is an individually rational TCBO mechanism.*

Furthermore, it is straightforward to identify individually rational TCBO mechanisms. Referring to control rights at the empty submatching as the initial control rights, let us formulate the criterion for individual rationality as follows.

Proposition 2. *In house allocation and exchange problems, a TCBO mechanism is individually rational if and only if it may be represented by a consistent control rights structure in which each agent is given the initial ownership rights of all houses from his endowment.*

Proof of Proposition 2. To prove individual rationality of the above subclass of TCBO mechanisms, consider an agent i and assume that i owns at the empty submatching a house h from his endowment. Then R4 ensures that i owns h throughout the execution of the TCBO algorithm. Thus, the TCBO mechanism will allocate to i house h or a house that i prefers to h.
Now, let ψ be an individually rational TCBO mechanism. Recall that ownership* was defined in the proof of Theorem 2. For any agent i and house h from i's endowment, i is owner* of h because individual rationality implies that $\psi[\succ](i) = h$ for any $\succ \in \mathbf{P}[\varnothing, h]$. The construction from the proof of Theorem 2 thus yields a control right structure that assigns to each agent the initial ownership rights over the houses form his endowment, and represents ψ. **QED**
Notice, that when one agent is endowed with all houses, there are individually rational mechanisms that might be represented both by a control right structure that assigns this agent initial ownership right over all houses, and by an alternative control right structure that assigns this agent ownership rights over all houses but one. Except for such situations however, any control right structure of an individually rational TCBO mechanism assigns to each agent the initial ownership rights of all houses from his endowment.

As a corollary of the above two results, we obtain a powerful and non-trivial characterization for an important subdomain of allocation and exchange problems:

Theorem 4. *In house allocation and exchange problems where each agent has a nonempty endowment, a mechanism is* individually rational*,* Pareto efficient*, and* group strategy-proof *if and only if it is a TTC mechanism (aka hierarchical exchange) that assigns all agents the initial ownership rights of houses from their endowment.*

Proof of Theorem 4. By Corollary 3, a mechanism φ is individually rational, Pareto-efficient and group strategy-proof if and only if there exists an individually rational and consistent control right structure (c, b) such that $\varphi = \psi^{c,b}$. By Proposition 2 we may assume that each agent has initial ownership rights over the houses from their endowment. By condition R4 of consistency all unmatched agents own a house throughout the mechanism, and hence R3 implies that no agent is a broker. $\psi^{c,b}$ is thus a TTC mechanism. **QED**

This result is a generalization of the result stated by Ma (1994) for the housing market of Shapley and Scarf (1974). A housing market is a house allocation and exchange problem in which $|I| = |H|$ and each agent is endowed with a house. In this environment, Ma characterized TTC (in which agents own their endowments) as the unique mechanism that is individually rational, strategy-proof, and Pareto efficient.

6.3 Market Design Environments

The assumptions of Theorem 3 are satisfied by the *house allocation problem with existing tenants* of Abdulkadiroğlu and Sönmez [1999]. Theirs is the subclass of house allocation and exchange problems in which each agent is endowed with one or zero houses. In the former case, the agent is referred to as existing tenant. The house allocation problem with existing tenants is modeled after the real-life dormitory allocation problems in the US college campuses. In each such college, at the beginning of the academic year, there are new senior, junior, sophomore students, each of whom already occupies a room from the last academic year. There are vacated rooms by the graduating class and there are new freshmen who would like to obtain a room, though they do not currently occupy any.

The assumptions of Theorem 4 are satisfied by the *kidney exchange* with strict preferences [Roth et al., 2004], and the *kidney exchange problem with good Samaritan donors* [Sönmez and Ünver, 2006]. Kidney transplant patients are the agents and live kidney donors are the houses. Each agent is endowed with a live donor, who would like to donate a kidney if his paired-donor receives a transplant in return. Thus, all agents have nonempty endowments. The model also allows for unattached donors known as good Samaritan donors who would like to donate a kidney to any patient. In the US, good Samaritan donors have been the driving force behind kidney exchange since 2006. Many regional programs such as Alliance for Paired Donation (centered in Toledo, Ohio) and New England Program for Kidney Exchange (centered in Newton, Massachusetts) have used good Samaritan donors in majority of kidney exchanges that they conducted since 2006 [cf. Rees et al., 2009].

The kidney exchange context underscores the importance of group strategy-proofness. The doctors of patients are the ones who have the information about patients' preferences over kidneys and it is known that doctors (or transplant centers) themselves manipulate the system, if it will benefit their patients.[12] An individually strategy-proof mechanism which is not group strategy-proof could thus by manipulated by the doctors. Group strategy-proofness guarantees that no doctor is able to manipulate the mechanism on behalf of his or her patients without hurting at least one of them.

7 Outside Options

In this final section, we drop the assumption that $|H| \geq |I|$ and allow agents to prefer their (non-tradeable) outside options to some of the houses. Thus, some agents may be matched with their outside options, and we need to slightly modify some of the definitions. As before I is the set of agents and H is the set of houses. Each agent i has a strict preference relation $\succ_i$ over H and his outside option, denoted y_i. We denote the set of outside options by Y. The houses preferred to the outside option are called acceptable (to the agent); the remaining houses are called unacceptable to this agent. As before we denote by $\mathbf{P}_i$ the set of agent i's preference profiles, and $\mathbf{P}_J = \times_{i \in J} \mathbf{P}_i$ for any $J \subseteq I$.

Let us initially restrict the attention to house allocation problems. This restriction can be easily relaxed as in Section 6, and we do it at the end of the section. As before, a house allocation problem is the triple $\langle I, H, \succ \rangle$. We impose no assumption on cardinalities of I and H, in particular we allow both $|H| \geq |I|$ and $|H| < |I|$.

We generalize the concept of submatching as follows. For $J \subseteq I$, a submatching is a one-to-one function $\sigma : J \to H \cup Y$ such that each agent is matched with a house or his outside option.

A terminological warning is in order. A natural interpretation of the outside option is remaining unmatched. We will not refer to the outside option in this way however in order to avoid confusion

[12] Deceased-donor queue procedures are gamed by physicians acting as advocates for their patients. In particular, in 2003 two Chicago hospitals settled a Federal lawsuit alleging that some patients had been fraudulently certified as sicker than they were to move them up on the liver transplant queue [Warmbir, 2003].

with our submatching terminology. As in the main body of the paper, whenever we say that an agent is unmatched at σ, we refer to agents from $\bar{I}_\sigma = I - I_\sigma$. An agent is considered matched even if he is matched to his outside option.

As before $\mathcal{S}$ is the set of submatchings, I_σ denotes the set of agents matched by σ, and $H_\sigma \subseteq H$ denote the set of houses matched by σ, and we use the standard function notation so that $\sigma(i)$ is the assignment of agent $i \in I_\sigma$, $\sigma^{-1}(h)$ is the agent that got house $h \in \sigma(I_\sigma)$, and $\sigma^{-1}(Y)$ is the set of agents matched to their outside options. A matching is a maximal submatching, that is $\mu \in \mathcal{S}$ is a matching if $I_\mu = I$. As before $\mathcal{M} \subset \mathcal{S}$ is the set of matchings. A (direct) mechanism is a mapping $\varphi : \mathbf{P} \longrightarrow \mathcal{M}$ that assigns a matching for each preference profile (or, equivalently, allocation problem). Mechanisms, efficiency, and group strategy-proofness are defined as before.

The control right structures (c, b) and their consistency R1-R6 are defined in the same words as before (notice though that the meaning of the words such as submatching has changed as explained above). In particular, (i) only houses are owned or brokered; the outside options are not, and (ii) the control rights are defined for all submatchings, including submatchings in which some agents are matched with their outside options. Notice that if a control right structure is consistent on the domain with outside options, and $|H| \geq |I|$, then the restriction of the control right structure to submatchings in which all agents are matched with houses is a consistent control right structure in the sense of Sections 4-5.

We will adjust the definition of the TCBO algorithm by adding two clauses.

Clause (a). We add the following provision to Step 1 (pointing) of round r:

- If an agent prefers his outside option to all unmatched houses, the agent points to the outside option. If there is a broker for whom the brokered house is the only acceptable house; such broker also points to his outside option. The outside option of each agent points to the agent.

- We modify the definition of σ^r in Step 3 (matching); σ^r is defined as the union of σ^{r-1} and of the set of agent-house pairs and agent-outside option pairs matched in Step 3.

Clause (b). In Step 3, we do not match agents in the cycle containing the broker except if leaving this cycle unmatched implies that no cycle is matched in the current round.

Clause (a) accommodates outside options. Clause (b) is added to make sure that we do not match a broker with his outside option when he prefers the brokered house to the outside option and the brokered house is not allocated to any other agent. Notice that the broker is matched only if any pointing sequence that starts with an owner ends by pointing to the broker.

We will refer to the mechanism of Section 4 modified by clauses (a) and (b) as outside options TCBO, and when there is no risk of confusion, simply as TCBO. We will refer to the mechanism $\psi^{c,b}$ resulting from running the outside options TCBO on consistent control right structure as outside options TCBO, or TCBO. Using the same name is justified because the mechanism described above can be used to allocate houses in the setting of Sections 2-6, and – when restricted to the case of $|H| \geq |I|$ and the subdomain of preferences in which all agents prefer any house to their outside option in the setting – is identical with the TCBO mechanism of Section 5. Indeed, in the restricted setting clause (a) is never invoked, and presence or absence of clause (b) has no impact on the allocation. This follows from the group strategy-proofness of TCBO of Section 5. Given a profile of agents' preferences, agents who are brokers along the run of the TCBO without clause (b) can replicate the run of TCBO with clause (b) as follows. The first agent who becomes a broker along the path of the algorithm, reports all houses which are matched in cycles not involving the broker ahead of the house the broker will be allocated, while keeping his preference profile otherwise intact. If another agent becomes a broker after the first broker is matched or looses his brokerage right, we modify this agent preferences in the same way, and same for other brokers. Were the outcomes of the mechanism dependent on whether the brokers simulate clause (b) or not, there would be a preference profile in which one of the brokers could either improve his outcome or bossy other agents contrary to group strategy-proofness. By Theorem 1 this is not possible. The fact that clause (b) does not impact allocation in the setting without outside option is analogous to the well known fact that in TTC the order in which we match cycles of agents does not matter.

In the presence of outside options, the TCBO class of mechanisms again coincides with the class of Pareto-efficient and group strategy-proof direct mechanisms. The proof resembles the proofs of Theorems 1 and 2; the required modifications are discussed in Appendix D.

Theorem 5. *In the environment with outside options, every TCBO mechanism is group strategy-proof and Pareto efficient. Moreover, every group strategy-proof and Pareto-efficient direct mechanism is TCBO.*

We are now ready to extend the characterization to the general allocation and exchange setting with outside options. As in Section 6, the social endowment $H_0 \subset H$ and agents' endowments $H_i \subset H$, $i \in I$, are disjoint and sum up to H. A house allocation and exchange problem is a list $\langle H, I, \mathcal{H}, \succ \rangle$ where $\mathcal{H} = \{H_i\}_{i \in \{0\} \cup I}$. The results of Section 6 translates to the setting with outside options; the proofs rely on Theorem 5 instead of Theorems 1 and 2, and are otherwise unchanged.

Theorem 6. *In house allocation and exchange problems with outside options, a mechanism is* individually rational*,* Pareto efficient*, and* group strategy-proof *if and only if it is an individually rational TCBO mechanism.*

As before, it is straightforward to identify individually rational TCBO mechanisms.

Proposition 3. *In house allocation and exchange problems with outside options, a TCBO mechanism is individually rational if and only if it may be represented by a consistent control rights structure in which each agent is given the initial ownership rights of all houses from his endowment.*

In the environment with outside options, we define the TTC mechanisms as TCBO with no brokers. Our characterization of TTC remains true.

Theorem 7. *In house allocation and exchange problems with outside options, if each agent has a nonempty endowment, then a mechanism is* individually rational*,* Pareto efficient*, and* group strategy-proof *if and only if it is a TTC mechanism (with outside options) that assigns all agents the initial ownership rights of houses from their endowment.*

A Appendix: Comments on Consistency Requirements

This appendix explains the consistency requirements R3, R5, and R6. The remaining requirements, R1, R2, and R4, were discussed in the main text.

R3 postulates that a broker does not own any houses. Dropping this assumption would violate efficiency. For instance, consider the case of two agents 1 and 2 such that agent 1 brokers house h_1 and owns house h_2 while 2 has no control rights. If agent 1 prefers h_1 over h_2 while agent 2 prefers h_2 over h_1 then running the TCBO algorithm (with the above inconsistent control right structure) would allocate h_2 to agent 1 and h_1 to agent 2 which is inefficient.

R5 might be called limited persistence of brokerage, and is the counterpart of R4 for brokers. R5 states that a brokerage right persist when we move from smaller to larger submatching provided two or more owners from the smaller submatching remain unmatched at the larger submatching. The following example illustrates why we need this requirement to keep TCBO individually strategy-proof:

Example 1. ***Why do we need R5 to prevent individual manipulation?*** Consider four agents $i_1, ..., i_4$. Assume that at the empty submatching agent i_2 brokers a house and other agents own one house each. Denote by h_k the house controlled by agent i_k. Let us maintain R1-R3, R4 and R6, and violate R5 by assuming that h_2 is owned by i_4 at submatching $\{(i_1, h_1)\}$. Now, there are two previous owners unmatched at $\{(i_1, h_1)\}$, i_3 and i_4. Moreover, i_2 is no longer a broker. Consider now a preference profile such that h_1 is i_1's and i_2's mutual first choice house, h_2 is the first choice of other agents, and h_3 is the second choice of i_2 and i_3. Under this preference profile and control rights structure, i_2 would benefit by misrepresenting his preferences and declaring h_3 to be his first choice.

R6 refers to the case where a broker loses his right at a submatching at which only a single previous owner is unmatched. In this case, the broker requires some protection against losing his right. That is to say, when the previous owner gets matched with the ex-brokered house, the ex-broker owns the houses of this owner. This is the *broker-to-heir transition* of the ex-broker.

The following two examples illustrate why we need R6 to keep TCBO both individually strategy-proof and non-bossy. The first one is similar to the above one:

Example 2. ***Why do we need R6 to prevent individual manipulation?*** Consider four agents $i_1, ..., i_4$. Assume that at the empty submatching agent i_2 brokers h_2, i_1 owns h_1, h_4, and i_3 owns h_3. At submatching $\{(i_1, h_1)\}$, assume that i_3 owns h_2 as well, and i_2 loses his brokerage right. Now, i_4 inherits h_4 as an owner. We assume R1-R5, while violate R6. R5 is not violated as there is a single previous owner unmatched at $\{(i_1, h_1)\}$, and he is i_3. However, R6 is violated as at the submatching $\{(i_1, h_1), (i_3, h_2\}$, i_2 is not the heir to i_3. That is, i_2 does not own ex-owned house h_3 of i_3, but i_4 does. Consider the preference profile at which, agents i_1 and i_2 have house h_1, i_2 has h_2 and i_4 has h_3 as their first choices; and agent i_2's second choice is h_3. Then, i_2 would benefit by ranking h_3 first.

Example 3. ***Why do we need R6 to prevent bossiness?*** Consider the same control rights structure as in Example 2. Consider the preference profile at which i_1 and i_3's first choices are h_1, and i_2 and i_4's first choice is h_3; second choice of i_3 is h_2. Agent i_3 will be bossy by ranking h_2 first. In both cases he receives house h_2. However, in the first case, i_2 receives h_4; while in the latter, he receives h_3.

Our last example shows that the complexity of R5-R6, while likely second order in terms of applications of the TCBO mechanisms, is inherent to the full class of group strategy proof and efficient mechanisms.

Example 4. ***Why R5 and R6 cannot be strengthened (consistent control rights with broker-to-heir transition).*** In the remark below we show that the following TCBO mechanism does not satisfy a natural strengthening (and simplification) of R5-R6. Consider an environment with four agents, i_1, i_2, i_3, i_4, four houses, h_1, h_2, h_3, h_4, and a TCBO mechanism $\psi^{c,b}$ whose control rights structure (c, b) is illustrated by the table in Figure 1 and is explained below.

Houses h_1, h_3 are owned by agent i_1 (denoted by "o" next to i_1 in the figure; when he is matched the unmatched of the two houses is owned by i_3 (if he is still unmatched), then i_2, and i_4 (in each

$(\mathbf{c}, \mathbf{b})$

$\mathbf{h_1}$		$\mathbf{h_2}$		$\mathbf{h_3}$		$\mathbf{h_4}$	
i_1,o		i_2,o		i_1,o		i_4,b	
i_3,o	$(i_1, h_1)(i_2, h_4)$ ↙		↘ otherwise	i_3,o	(i_1, h_1) ↙		↘ $(i_1, x)(i_2, y)(i_3, z)$ $x \neq h_1$
i_2,o	i_4,o		i_1,o	i_2,o	i_2,o		i_4,o
i_4,o	i_3,o		i_3,o	i_4,o	i_4,o		
			i_4,o		i_3,o		

Figure 1: A control rights structure with broker-to-heir transition

of these cases, and in the below ownership situations, as the owner remains as an owner until he is matched, R1 is satisfied).

House h_2 is owned by i_2, and unless i_1 is matched with h_1 and i_2 is matched with h_4 then next owners of h_2 are i_1, i_3, and i_4, in order. The other case will be explained after we explain the control rights of h_4.

Agent i_4 has the brokerage right over h_4 initially (i.e., at the empty submatching, denoted by "b" next to i_4 in the figure). He remains as the broker as long as he is unmatched except the cases that i_1 is matched with h_1 and he is the only remaining unmatched agent. In the prior case, we have a broker-to-heir transition (as all houses are owned initially by i_1 or i_2, at this point i_2 is the only remaining owner left from the previous round, hence R6 is satisfied). Then, agent i_2 becomes the owner of h_4, and then i_4 and i_3 own h_4, in order. The above transition has an implication on control right structure of h_2 (by R6). When i_2 gets matched with the ex-brokered house, h_4, then i_4, the ex-broker, who is now heir to i_2 owns h_2, the sole unmatched and previously owned house of i_2. In the second case, as the broker is the only agent left unmatched, he owns all existing houses, in this case it is h_4 (thus, R2 is satisfied).

Notice, that R3 is satisfied as broker i_4 is the last inheritor of all owned houses unless he exits brokerage.

Remark 1. *The TCBO mechanism defined by the consistent control rights structure in Example 4, (c, b), is different from all TCBO mechanisms with consistent control right structures in which the simple analogue of R4 for brokers holds true: "if $|\sigma'| < |I| - 1$ and agent i brokers house h at σ and is unmatched at $\sigma' \supset \sigma$, then i brokers h at σ' ."*

Proof of Remark 1. By way of contradiction, let us assume that there is a TCBO mechanism ψ with control right structure satisfying the above strong form of brokerage persistence and that produces the same allocation as $\psi^{c,b}$ for each profile of agents' preferences.

First, notice that at the empty submatching, i_4 is the *broker* of h_4 in ψ. This is so, because h_4 is not owned by any agent at the empty submatching $\varnothing$ as $(\psi[\succ])^{-1}(h_4) = (\psi^{c,b}[\succ])^{-1}(h_4)$ varies with $\succ \in \mathbf{P}$ (that is across profiles at which all agents rank h_4 first). Hence, there is an agent who has the brokerage right over h_4 and it must be i_4 as $\psi[\succ](i_4) = \psi^{c,b}[\succ](i_4) = g$ for all $\succ \in \mathbf{P}$ such that all agents rank h_4 first and any $g \in \{h_1, h_3, h_2\}$ second.

Second, consider the submatching $\sigma = \{(i_1, h_1)\}$ and a preference profile $\succ \in \mathbf{P}$ such that i_1 ranks h_1 first, others rank h_4, h_3, h_2, and h_1 in this order. In mechanism ψ, agent i_4 would continue to be the broker of h_4 at σ and thus

$$\psi[\succ](i_4) = h_3.$$

However,

$$\psi^{c,b}[\succ](i_4) = h_2.$$

This contradiction concludes the proof. **QED**

B Appendix: Proof of Theorem 1

Proof of Theorem 1. In the main text we showed that TCBO is Pareto efficient and individually strategy-proof. By Papai's Lemma 1, it is enough to show that TCBO is non-bossy. Let $\psi^{c,b}$ be a TCBO mechanism. Fix an agent $i_* \in I$ and two preference profiles $\succ = [\succ_{i_*}, \succ_{-i_*}]$ and $\succ' = [\succ'_{i_*}, \succ_{-i_*}]$ such that

$$h_* = \psi^{c,b}[\succ'](i_*) = \psi^{c,b}[\succ](i_*).$$

Let s be the round i_* leaves (with house h_*) submitting $\succ_{i_*}$ and s' be the time i_* leaves (with h_*) submitting $\succ'_{i_*}$. By symmetry, it is enough to consider the case $s \leq s'$. In order to show that

$$\psi^{c,b}[\succ](i) = \psi^{c,b}[\succ'](i) \qquad \forall\, i \in I,$$

we will prove the following stronger statement:

Hypothesis: If a cycle of agents $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms and is removed at round r when preferences $\succ$ were submitted, then either

- same cycle $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms when preferences $\succ'$ are submitted, or
- $n = 2$ and two cycles $h^2 \to i^1 \to h^2$ and $h^1 \to i^2 \to h^1$ form when preferences $\succ'$ are submitted, or

- $n=1$ and there exists agent $j \neq i^1$ and house $h \neq h^1$ such that the cycle $h \to i^1 \to h^1 \to j \to h$ forms when preferences $\succ'$ are submitted.

Whenever in the proof we encounter cycles of length n, the superscripts on houses and agents will be understood to be modulo n, that is $i^{n+1} = i^1$ and $h^{n+1} = h^1$.

By Lemma 3, the above hypothesis is true for all $r < s$. The proof for $r \geq s$ will proceed by induction over the round r.

Initial step. Consider $r = s$. Under $\succ$, house h_*^1 points to agent $i_* = i_*^1$ points to house $h_* = h_*^2$ that points to agent i_*^2 that points to ... that agent i_*^n that points to house h_*^1, and the cycle

$$h_*^1 \to i_*^1 \to h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1$$

is removed at round s. Lemma 3 implies that the same houses and agents are in the market at time s under both $\succ$ and $\succ'$ and that all agents from $I_{\sigma^s[\succ]} - \{i_*^1, ..., i_*^n\}$ are matched by $\sigma^s[\succ']$ in the same way as in $\sigma^s[\succ]$.

Lemma 3 also implies that the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ forms at round s under preferences $\succ'$.

If all pairs (i_*^ℓ, h_*^ℓ), for all $\ell \in \{2, ..., n\}$, are $\sigma^s[\succ]$-o-pairs, then they are $\sigma^s[\succ']$-o-pairs and the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ will stay in the system as long as i_*^1 is in the system (by persistency of o-pairs through R4). Thus, at s' all agents $i_*^1, ..., i_*^n$ would leave with same houses as under $\succ$.

If $n > 1$, and (i_*^ℓ, h_*^ℓ) is a broker-brokered house pair for some $\ell \in \{2, ..., n\}$, then the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ will stay in the system as long as (i_*^ℓ, h_*^ℓ) continues to be a broker-brokered house pair. If (i_*^ℓ, h_*^ℓ) continues to be a broker-brokered house pair till round s' under $\succeq$, then the initial step is proved. Otherwise, there is a round $s'' \in \{s+1, ..., s'\}$ such that agent i_*^ℓ has brokerage right over h_*^ℓ at rounds $s, ..., s''-1$ but not at round s''. By R6's broker-to-heir transition property, $n = 2$ and $i_*^{\ell+1}$ owns h_*^ℓ at $\sigma^{s''}[\succ']$ because he owns $h_*^{\ell+1}$at both $\sigma^{s''-1}[\succ']$ and $\sigma^{s''}[\succ']$. As $i_*^{\ell+1}$ top preference is then h_*^ℓ, he will leave with it at s''. By R6's broker-to-heir transition property, agent i_*^ℓ will inherit $h_*^{\ell+1}$ at $s''+1$ and will be matched with it. This case ends the proof of the the inductive hypothesis for $r = s$.

Inductive step. Now, take any round $r > s$ such that $\sigma^r[\succ] - \sigma^{r-1}[\succ]$ is non-empty, and assume that the inductive hypothesis is true for all rounds up to $r-1$. Consider agents and houses

$$h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$$

that form a cycle at round r under $\succ$. Since all agents but i^* have same preferences in both profiles $\succ$ and $\succ'$, so do agents $i^1, ..., i^n$.

Let r' be the earliest round in which one of these agents is matched under $\succ'$, and let i^ℓ be an agent matched at r'. Recall that $\psi^{c,b}[\succ](i^\ell) = h^{\ell+1}$. The argument will be divided into ten claims, the first of which follows directly from the inductive hypothesis.

Claim 1. Suppose that a house $h \in \sigma^{r-1}[\succ]$ forms a cycle with at least two agents under $\succ$. Then:

- If agent $i \in I$ belongs to the cycle of house h under $\succ'$, then i belongs to the cycle of h under $\succ$.
- If house h' belongs to the cycle of h under $\succ'$, then h' belongs to the cycle of h under $\succ$.

Claim 2. Suppose $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$ and agent i owns house h at ν, such that h belongs to a cycle at some round r'' under $\succ'$ and at round r under $\succ$. If j controls h at $\sigma^{r''-1}[\succ']$ and is unmatched at ν, then i is in one cycle with h at round r'' under $\succ'$.

Proof of Claim 2: If $i = j$, then the result is true. Assume $i \neq j$. Then j does not control h at ν. Thus, j brokers it at $\sigma^{r''-1}[\succ']$ and hence the cycle of h at round r'' under $\succ'$ contains some other agent j' and house h' that j' owns. By Claim 1, $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, and thus $j' \notin I_\nu$ and $h' \notin H_\nu$. By R4, R5, and R6, j' owns h at ν, and thus $i = j'$. QED

Claim 3. Suppose that $n > 1$ and agents $j, i^1, ..., i^n$, and house h^1 are unmatched at $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, such that j and $h^{\ell+1}$ are part of a cycle matched at some round $r'' \leq r'$ under $\succ'$. If agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, then, under $\succ'$, $i^{\ell+1}$ and $h^{\ell+1}$ are both matched at round r'.

Proof of Claim 3: If j owns $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$ then R4 implies that both j and $i^{\ell+1}$ own $h^{\ell+1}$ at ν. Hence, $i^{\ell+1} = j$, and he is matched at r'' under $\succ'$. Because r' is the earliest round one of the agents $i^1, ..., i^n$ is matched, it must be that $r' = r''$ and the claim is true.

If j brokers $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, then let σ_0 be a minimal submatching in

$$\left\{\sigma \in \mathcal{S} : \sigma^{r''-1}[\succ'] \subseteq \sigma \subseteq \nu\right\}$$

at which j is not a broker of $h^{\ell+1}$. Let $j' \neq j$ belong to the cycle of $h^{\ell+1}$ at round r'' under $\succ'$. Then j' is an owner of a house h' at $\sigma^{r''-1}[\succ']$. Because $n > 1$, Claim 1 gives $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$. Thus, (j', h') is an owner-owned house pair at $\sigma_0 \subseteq \nu$. By R5 and R6, agent j' becomes the owner of $h^{\ell+1}$ at σ_0, and thus by R4, i.e., persistence of an ownership, $(j', h^{\ell+1})$ is an owner-owned house pair at ν. Thus, $j' = i^{\ell+1}$ and he is matched at r'' in the same cycle as $h^{\ell+1}$ under $\succ'$. A fortiori, $r'' = r'$. QED

Claim 4. Assume i^ℓ is a $\sigma^{r-1}[\succ]$-owner. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round then they are matched with each other.

Proof of Claim 4: House $h^{\ell+1}$ is agent i^ℓ's top choice among houses unmatched at $\sigma^{r-1}[\succ]$. By the inductive assumption, all houses matched before round r under $\succ$ are also matched with the same agents under $\succ$. Since $h^{\ell+1} = \psi^{c,b}[\succ](i^\ell)$ is the top choice of houses remaining at round r under $\succ$, $\psi^{c,b}[\succ'](i^\ell)$ is weakly worse than $h^{\ell+1}$. Hence, at the round i^ℓ is matched he points to $h^{\ell+1}$ (and then is matched with it) or a worse house (and then $h^{\ell+1}$ was matched earlier). QED

Claim 5. Suppose that agents $j, i^1, ..., i^n$, and house $h^{\ell+1}$ are unmatched at $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, $i^{\ell+1}$ brokers $h^{\ell+1}$ at $\sigma^{r-1}[\succ]$, agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, such that agent j and house $h^{\ell+1}$ are part of a cycle matched at some round $r'' \leq r'$ under $\succ'$. Then, house $h^{\ell+1}$ is matched at round $r' = r''$ under $\succ'$.

Claim 6. Furthermore, the inductive hypothesis is true for round r or $i^{\ell+1}$ is matched at round r' under $\succ'$.

Proofs of Claim 5 and Claim 6: We will first prove Claim 5 and continue from that point on proving Claim 6. For convenience, assume $\ell = n$ that is $\ell + 1 = 1$. Since, i^1 is a broker when he was matched under $\succ$, $n > 1$, and thus, we can use Claim 1. Moreover, if $i^1 = j$, then i^1 controls h^1 at $\sigma^{r''-1}[\succ']$, and thus he is matched at round r'' under $\succ'$. Because r' is the earliest round one of the agents $i^1, ..., i^n$ is matched, it must be that $r' = r''$ and the claim is true. Hence, assume that $i^1 \neq j$.

Notice that

- If j controls h^1 at ν then i^1 does not.
- If j does not control h^1 at ν then j brokers it at $\sigma^{r''-1}[\succ']$, and thus, the cycle of h^1 contains some $\sigma^{r''-1}[\succ']$-owner j' and some house h' owned by j'. By Claim 1, $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, and thus j' and h' are not matched at ν. By R5 and R6, if j stops brokering h^1, then j' owns h^1 at ν, and hence i^1 cannot broker it at ν.

In either case, i^1 does not broker h^1 at ν while he brokers it at $\sigma^{r-1}[\succ]$. Thus, R5 and R6 imply that

$$n = 2, i^2 \text{ owns } h^1 \text{ at } \nu, \text{ and } i^1 \text{will own } h^2 \text{ if } i^2 \text{is matched with } h^1 \text{ at } \nu. \tag{1}$$

If $i^2(=i^\ell) \neq j$, then Claim 2 implies that i^2 is in the cycle of h^1 that forms at round r'' under $\succ'$, and thus $r' = r''$.

If $i^2(=i^\ell) = j$, Claim 4 implies that i^2 is matched at r'' under $\succ'$ in the cycle

$$h^1 \to i^2 \to h^1.$$

Thus, $r'' = r'$ and $h^1(=h^{\ell+1})$ is matched at round r' under $\succ'$. That proves Claim 5.

In order to prove Claim 6, let r^1 be the time i^1 is matched, and r^2 be the time h^2 is matched under $\succ'$. By Claim 7 (whose proof depends on Claim 5 but not on Claim 6), $r^2 \leq r^1$. Let j^2 be the agent controlling h^2 at $\sigma^{r^2-1}[\succ']$. By Claim 1, $j^2 \notin I_{\sigma^{r-1}[\succ]}$. Let

$$\nu' = \sigma^{r-1}[\succ] \cup \sigma^{r^2-1}[\succ'].$$

If $r^2 > r'$ then i^2 is matched to h^1 at $\nu' \supseteq \sigma^{r^2-1}[\succ'] \supseteq \sigma^{r'}[\succ']$. By Statement in (1), i^1 owns h^2 at ν'. By Claim 2, i^1 is in one cycle with h^2 at r^2.

If $r^2 \leq r'$ then i^2, the $\sigma^{r-1}[\succ]$-owner of h^2, is unmatched at $\sigma^{r^2-1}[\succ']$ and hence, owns h^2 at ν'. By Claim 2, $i^2(=i^\ell)$ is in one cycle with $h^2(=h^\ell)$ at r^2 under $\succ'$. That means that $r^2 = r'$. Let i be the agent controlling h^1 at $\sigma^{r'-1}[\succ']$. Since the cycle of h^1 contains i^1 and i^2, and i^2 gets h^1, hence $i \neq i^2$. Because, i^2 owns h^1 and h^2 at ν', R5 and R6 imply that i inherits h^2 if i^2 is matched with h^1 at ν'. By Statement in (1), i^1 inherits h^2 if i^2 is matched with h^1 at ν, and hence also at ν'. Thus, $i^2(=i^\ell) = i$ belongs to the cycle of $h^1(=h^{\ell+1})$ under $\succ'$. QED

Claim 7. Assume i^ℓ is a $\sigma^{r-1}[\succ]$-broker. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round, then they are matched with each other.

Proof of Claim 7: Notice that $n > 1$ and for notational convenience assume that $\ell = 1$. By the inductive assumption, There exists r^* such that $\sigma^{r-1}[\succ] \subseteq \sigma^{r^*}[\succ']$. Thus, if (i^1, h^2) does not satisfy the claim then the top preference of i^1 must be h^1, the house he brokers at $\sigma^{r-1}[\succ]$, and i^1 must get h^1 under $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r'-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ and house h^1 are unmatched at the submatching $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. By Claim 5, h^1 is matched at r', and by Claim 4, i^n gets h^1, a contradiction. QED

Claim 8. If $n = 1$ then either $h^1 \to i^1 \to h^1$ form a cycle under $\succ'$, or there exists an agent j and a house h such that $h \to i^1 \to h^1 \to j \to h$ form a cycle under $\succ'$.

Proof of Claim 8: Claim 2 implies that h^1 is matched at round $r'' \leq r'$ when preferences are $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Two cases are possible about j:

Case 1. $j \in I_{\sigma^{r-1}[\succ]}$: Then $j \neq i^1$ and the inductive assumption and $h^1 \notin H_{\sigma^{r-1}[\succ]}$ imply that

- j is matched at round r under $\succ$ in a cycle $h \to j \to h$ (for some house $h \neq h^1$), and
- there exists agent i such that the cycle $h \to i \to h^1 \to j \to h$ is matched at round r'' under $\succ'$.

To finish the proof of the current case, it remains to be shown that $i = i^1$.

- If $i \in I_{\sigma^{r-1}[\succ]}$ then the inductive assumption implies that i is matched with h^1 under $\succ'$, and hence $i = i^1$.
- If $i \notin I_{\sigma^{r-1}[\succ]}$ then i and i^1 are unmatched at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and by R4, i.e., persistence of o-pairs, i^1 owns h^1 at this submatching. Notice that j is an owner of h at $\sigma^{r-2}[\succ]$, and hence at $\sigma^{r-2}[\succ] \cup \sigma^{r''-1}[\succ']$. Thus, i must have been a broker of h at $\sigma^{r''-1}[\succ']$ and stopped being a broker at a submatching σ between $\sigma^{r''-1}[\succ']$ and $\sigma^{r-2}[\succ] \cup \sigma^{r''-1}[\succ']$. Because there might be only one broker at each submatching, j is an owner of h^1 at $\sigma^{r''-1}[\succ']$. Thus j is an owner at σ, and inherits h when i loses the broker status. When j is matched with h, the broker-to-heir transition rule R6 implies that i becomes the owner of h^1. Hence, i is the owner of h^1 at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$ as is i^1. Thus, $i = i^1$.

Case 2. $j \notin I_{\sigma^{r-1}[\succ]}$: Then, agents j, i^1 are unmatched at the submatching $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and by R4, i.e., persistence of o-pairs, i^1 owns h^1 at this submatching. Hence, either

- $j = i^1$ controls h^1 at $\sigma^{r''-1}[\succ']$, or
- $j \neq i^1$ is a broker of h^1 at $\sigma^{r''-1}[\succ']$, and loses the brokerage right at some submatching σ between $\sigma^{r''-1}[\succ']$ and $\sigma^{r-1}[\succ] \cup \sigma^{r''}[\succ']$.

In the former subcase, i^1 is matched at r'' as h^1 is matched under $\succ'$. Thus, $r'' = r'$, and a fortiori i^1 is matched with $h^1 (= i^\ell)$ and hence owns h^1 at r'. The claim is then proved.

In the latter subcase, let $j' \neq j$ be an agent matched in the same cycle as h^1 at round r'' under $\succ'$. Then j' is an owner of a house h' at $\sigma^{r''-1}[\succ']$.

We have $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, as otherwise the inductive assumption and $h^1 \notin H_{\sigma^{r-1}[\succ]}$ would imply that j' is matched at $\sigma^{r-1}[\succ]$ before round r under $\succ$ in a cycle $h' \to j' \to h'$, and there would exist an agent i such that the cycle $h' \to i \to h^1 \to j' \to h'$ is matched at r'' under $\succ'$. The inductive assumption would further imply that h^1 is matched to i at $\succ$ and thus $i^1 = i$. Since j is in the cycle of h^1 and $j \neq j'$ and $j \neq i^1$, we would obtain a contradiction showing that $j' \notin I_{\sigma^{r-1}[\succ]}$.

Thus j' and h' are unmatched at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and hence they are unmatched at σ. Thus, R5 and R6's broker-to-heir transition property implies that agent j' is the owner of h^1 at σ, and by persistence of o-pairs through R4, (j', h^1) is an owner-owned house pair at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. Thus, $i^1 = j'$ and he is matched with h^1 at $r'' = r'$. Thus, at round r' under $\succ'$ the cycle in which i^1 is matched is $h' \to i^1 \to h^1 \to j \to h'$. QED

Claim 9. Suppose $n > 1$. If $i^{\ell+1}$ is $\sigma^{r-1}[\succ]$-owner, then i^ℓ is matched with $h^{\ell+1}$ at round r' under $\succ$, and $i^{\ell+1}$ is also matched at round r' under $\succ'$.

Proof of Claim 9: By Claim 2, house $h^{\ell+1}$ is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as $h^{\ell+1}$ at round r'' under $\succ'$. Since $h^{\ell+1} \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and Claim 3 yields that $i^{\ell+1}$ is matched at $r'' = r'$ and then Claim 2 shows that i^ℓ is matched with $h^{\ell+1}$. QED

Claim 10. If $i^{\ell+1}$ is a $\sigma^{r-1}[\succ]$-broker, then either the inductive hypothesis is true or i^ℓ is matched with $h^{\ell+1}$ at round r' under $\succ'$, and $i^{\ell+1}$ is also matched at round r' under $\succ'$.

Proof of Claim 10: For convenience let us assume that $\ell = n$ and $\ell + 1 = 1$. Agent i^n is a $\sigma^{r-1}[\succ]$-owner because $n > 1$ and i^1 is a $\sigma^{r-1}[\succ]$-broker. By Claim 2, house h^1 is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. Claim 5 yields $r'' = r'$ and thus Claim 4 shows that i^n is matched with h^1 under $\succ'$. Claim 6 ends the proof. QED

Claim 8 proves the inductive hypothesis for cycles of length $n = 1$ and Claim 9 and Claim 10 applied iteratively prove the hypothesis for cycles of length $n > 1$. This ends the proof of the theorem. **QED**

C Appendix: Proof of Theorem 2 (Implementation Result)

Let φ be a group strategy-proof and Pareto-efficient mechanism (fixed throughout the proof). We are to prove that φ may be represented as a TCBO mechanism. We will first construct the candidate control rights structure (c,b) and then show that the induced TCBO mechanism $\psi^{c,b}$ is equivalent to φ.

Let us start by introducing some useful terms and notation. Let $\sigma \in \overline{\mathcal{M}}$, $n \geq 0$ and $h^1, h^2, ..., h^n \in \overline{H_\sigma}$, and $i \in I$.

$\mathbf{P}_i[\sigma, h^1, ..., h^n]$ is the set of preferences $\succ_i$ of agent i such that

- if $i \in I_\sigma$, then

$$\sigma(i) \succ_i g \text{ for all } g \in H - \{\sigma(i)\},$$

- if $i \in \overline{I_\sigma}$, then

$$h^1 \succeq_i h^2 \succeq ... \succeq_i h^n \succ_i g \succ_i g' \text{ for all } g \in \overline{H_\sigma} - \left\{h^1, ..., h^n\right\} \text{ and all } g' \in H_\sigma.$$

That is, if i is not matched in submatching σ, $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ is the set of preferences that rank h^1,...,h^n in order over the remaining houses unmatched under σ, and rank those over the houses matched under σ; otherwise, $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ is the set of preferences that rank agent i's match under σ over all other houses (observe that $\mathbf{P}_i[\varnothing] \equiv \mathbf{P}_i$).

$\mathbf{P}[\sigma, h^1, ..., h^n] \subseteq \mathbf{P}$ is the Cartesian product of $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ over all $i \in I$. We define

$$\mathbf{P}^*[\sigma, h] = \cup_{h' \in \overline{H_\sigma} - \{h\}} \mathbf{P}[\sigma, h, h'],$$

i.e., the set of preference profiles generated by $\mathbf{P}[\sigma, h]$ that rank the same house as the second choice across all agents unmatched under σ.

When σ is fixed, we will occasionally write $\langle h^1, ..., h^n, ...\rangle$ instead of $\mathbf{P}_i[\sigma, h^1, ..., h^n]$.

We are ready to introduce some new terminology for the mechanism φ that is similar to the control rights structure terminology of the TCBO mechanisms. To distinguish the two classes defined for TCBO and φ, we will suffix these new definitions with *.

A house $h \in \overline{H_\sigma}$ is an **owned* house at** $\sigma \in \overline{\mathcal{M}}$ if $\varphi[\succ]^{-1}(h) = i$ for all $\succ \in \mathbf{P}[\sigma, h]$ for some $i \in \overline{I_\sigma}$; we refer to i as the **owner* of** h **at** σ.

A house $e \in \overline{H_\sigma}$ is a **brokered* house at** $\sigma \in \overline{\mathcal{M}}$ if there exist some $\succ$ and $\succ' \in \mathbf{P}^*[\sigma, e]$ such that $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$. Agent k is the **broker* of** e **at** σ if e is a brokered* house at σ and for all $\succ \in \mathbf{P}^*[\sigma, e]$ house $\varphi[\succ](k)$ is the second choice of k in $\succ_k$.[13]

[13]It may appear from the definitions that there is a third option for an unmatched house besides being owned* and brokered* at a submatching. However, Corollary 2 below show that these are the only two options.

Notice that if φ is a TCBO mechanism and i is an owner at σ then i is an owner* at σ, similarly for broker*. Thus, owners* and brokers* are the *candidate* owners and brokers in the TCBO mechanism that we will construct. We will show that the starred terms can be used to determine a consistent control rights structure (c, b) and a TCBO mechanism $\psi^{c,b}$. The proof of Theorem 2 will be finished after we show that $\varphi = \psi^{c,b}$.

Two lemmata proved in Pápai [2000] will be useful. Following her definition, we say that j **envies** i **at** $\succ$ if

$$\varphi[\succ](i) \succ_j \varphi[\succ](j).$$

Lemma 4. *(Pápai 2000) For all $i, j \in I$, all $\succ \in \mathbf{P}$, and all $\succ_j^* \in \mathbf{P}_j$, if j envies i at $\succ$ and $\varphi[\succ_j^*, \succ_{-j}](i) \neq \varphi[\succ](i)$, then*

$$\varphi[\succ](i) \succ_i \varphi[\succ_j^*, \succ_{-j}](i).$$

Lemma 5. *(Pápai 2000) For all $i, j \in I$, all $\succ \in \mathbf{P}$, and all $\succ_j^* \in \mathbf{P}_j$, if j envies i at $\succ$ and $\varphi[\succ_j^*, \succ_{-j}](i) \neq \varphi[\succ](i)$, then there exists $\succ_i^* \in \mathbf{P}_i$ such that*

$$\varphi[\succ_i^*, \succ_j^*, \succ_{-\{i,j\}}](i) = \varphi[\succ](j).$$

The following is an immediate corollary of strategy-proofness and Lemma 5:

Corollary 1. *For all $i, j \in I$, all $\succ \in \mathbf{P}$, and all $\succ_j^* \in \mathbf{P}_j$, if j envies i at $\succ$ and $\varphi[\succ_j^*, \succ_{-j}](i) \neq \varphi[\succ](i)$, then*

$$\varphi[\succ_j^*, \succ_{-j}](i) \succeq_i \varphi[\succ](j).$$

C.1 The Starred Control Right Structure is Well Defined

The lemma below show that if a house does not have a well-defined owner*, then it has a well-defined broker*. Thus the starred (candidate) control right structure is well defined. All lemmata in this section are formulated and proven at a fixed submatching $\sigma \in \overline{\mathcal{M}}$.

Lemma 6. *Let $\sigma \in \overline{\mathcal{M}}$. For all $i \in I_\sigma$ and all $h \in \overline{H_\sigma}$,*

$$\varphi[\succ](i) = \sigma(i) \text{ for all } \succ \in \mathbf{P}[\sigma, h].$$

Proof of Lemma 6. Suppose that an agent in $i \in I_\sigma$ does not get $\sigma(i)$ at $\varphi[\succ]$. Then we can create a matching by assigning all agents in $\overline{I_\sigma}$ that get a house in H_σ a house in $\overline{H_\sigma}$ that was assigned to an agent in I_σ, all other agents j in $\overline{I_\sigma}$ the house $\varphi[\succ](j)$, and all agents j in I_σ the house $\sigma(j)$. Since each agent in $\overline{I_\sigma}$ ranks houses in H_σ lower than houses in $\overline{H_\sigma}$ and each agent in I_σ ranks his σ-house as his first choice, this new matching Pareto dominates $\varphi[\succ]$, contradicting φ is Pareto efficient.**QED**

Lemma 7. *Let $\sigma \in \overline{\mathcal{M}}$ and $e, h \in \overline{H_\sigma}$. Then there exists some agent $i \in \overline{I_\sigma}$ such that $\varphi[\succ](i) = e$ for all $\succ \in \mathbf{P}[\sigma, e, h]$.*

Proof of Lemma 7 By way of contradiction suppose that $\succ, \succ' \in \mathbf{P}[\sigma, e, h]$ are such that $\varphi[\succ](i) = e$ and $\varphi[\succ'](i') = e$ for some $i' \neq i$.

Without loss of generality, we assume that $\succ$ and $\succ'$ differ only in preferences of a single agent $j \in \overline{I_\sigma}$. Let $g = \varphi[\succ](j)$ and $g\mathbf{P} = \varphi[\succ'](j)$. By non-bossiness, $g \neq g'$. By strategy-proofness, $j \notin \{i, i'\}$, and hence, $e \notin \{g, g'\}$. Moreover by Maskin monotonicity, if it were true that $g = h$, then $\varphi[\succ'] = \varphi[\succ]$ would be true, contradicting $\varphi[\succ'] \neq \varphi[\succ]$. Thus, $g \neq h$. Similarly $g' \neq h$. Moreover, by Maskin monotonicity for j, without loss of generality, we further assume that

$$\succ_j \in \langle e, h, g, g', ... \rangle \text{ and } \succ'_j \in \langle e, h, g', g, ... \rangle ,$$

and that the only difference between $\succ_j$ and $\succ'_j$ is in relative ranking of g and g'.

Let $k \neq i, i', j$ be some agent.

Define

$$\succ^*_k \in \langle e, g, h, ... \rangle ,$$
$$\succ^*_i \in \langle h, e, ... \rangle , \text{ and}$$
$$\succ^*_{i'} \in \langle h, e, ... \rangle .$$

where the relative ordering of all other houses coincide under $\succ^*_k, \succ^*_i, \succ^*_{i'}$ with that under $\succ_k, \succ_i, \succ_{i'}$, respectively.

We will first prove a number of claims.

Claim 1. (1) $\varphi[\succ^*_i, \succ_{-i}](i) = h$ and $\varphi[\succ^*_{i'}, \succ'_{-i'}](i') = h$.

(2) Moreover, if $\varphi[\succ](k) = h$ and $\varphi[\succ'](k) = h$, then $\varphi[\succ^*_i, \succ_{-i}](j) = g$.

Proof of Claim 1.

(1) By strategy-proofness for i, $\varphi[\succ^*_i, \succ_{-i}](i) \succeq^*_i e$. Everybody else in $\overline{I_\sigma}$ ranks e over h. Thus, by Lemma 6 and Pareto efficiency, i should get h at $[\succ^*_i, \succ_{-i}]$. The symmetric argument implies that $\varphi[\succ^*_{i'}, \succ'_{-i'}](i') = h$.

(2) Let $\varphi[\succ](k) = h$ and $\varphi[\succ'](k) = h$. By Maskin monotonicity regarding i, $\varphi[\succ^*_i, \succ_{-i}] = \varphi[\succ']$. Thus, j gets g' at $[\succ^*_i, \succ'_{-i}]$. By strategy-proofness for j, agent j gets at least g' and no house better

than g at $[\succ_i^*, \succ_{-i}]$ (recall that between $\succ_{-i}$ and $\succ'_{-i}$ only j changes his preferences). Suppose j gets g' at $[\succ_i^*, \succ_{-i}]$. Then, by Maskin monotonicity regarding j, we have $\varphi[\succ_i^*, \succ'_{-i}] = \varphi[\succ_i^*, \succ'_{-i}]$. In particular, $\varphi[\succ_{i'}^*, \succ_{-i'}](i') = \varphi[\succ_{i'}^*, \succ'_{-i'}](i') = h$. By Maskin monotonicity regarding i', $\varphi[\succ_{i'}^*, \succ_{-i'}] = \varphi[\succ]$. On the other hand, we have $\varphi[\succ](k) = h$, contradicting $\varphi[\succ_{i'}^*, \succ_{-i'}](i') = \varphi[\succ](i') = h$. Therefore, $\varphi[\succ_{i'}^*, \succ_{-i'}](j) = g'$. QED

Claim 2. If $\varphi[\succ](k) = h$, then $\varphi[\succ_k^*, \succ_{-k}](k) = g$ and $\varphi[\succ_k^*, \succ'_{-k}] = \varphi[\succ_k^*, \succ_{-k}]$.

Proof of Claim 2. Let $\varphi[\succ](k) = h$. Recall that $\succ_k^* \in \langle e, g, h, ...\rangle$. By strategy-proofness, since k gets h at $\succ$, agent k cannot get e and gets at least h at $[\succ_k^*, \succ_{-k}]$. Thus, k gets h or g at $[\succ_k^*, \succ_{-k}]$. Everybody else in $\overline{I_\sigma}$ ranks h over g. Thus, by Lemma 6 and Pareto efficiency, agent k should get g at $[\succ_k^*, \succ_{-k}]$.

These two profiles, $[\succ_k^*, \succ'_{-k}]$ and $[\succ_k^*, \succ_{-k}]$, only differ in preferences of agent j who ranks g above g' at $\succ_j$ and the other way at $\succ'_j$. We established above that j does not get g at $[\succ_k^*, \succ_{-k}]$. Maskin monotonicity regarding j implies $\varphi[\succ_k^*, \succ'_{-k}] = \varphi[\succ_k^*, \succ_{-k}]$. QED

Claim 3. If $\varphi[\succ](k) = h$, then $\varphi[\succ_k^*, \succ_{-k}](i) \in \{e, h\}$; furthermore if it is also true that $\varphi[\succ'](k) = h$, then $\{\varphi[\succ_k^*, \succ_{-k}](i), \varphi[\succ_k^*, \succ_{-k}](i')\} = \{e, h\}$.

Proof of Claim 3. Let $\varphi[\succ](k) = h$. Agent k envies agent i at $\succ$. Thus, by Corollary 1, agent i gets at least $h = \varphi[\succ](k)$ at $[\succ_k^*, \succ_{-k}]$. Hence $\varphi[\succ_k^*, \succ_{-k}](i) \in \{e, h\}$. By Claim 2, $\varphi[\succ_k^*, \succ'_{-k}](i) \in \{e, h\}$.

Also let $\varphi[\succ'](k) = h$. The symmetric argument as above using $\succ'$ instead of $\succ$ and i' instead of i shows that $\varphi[\succ_k^*, \succ'_{-k}](i') \in \{e, h\}$. Furthermore, Claim 2 implies that $\varphi[\succ_k^*, \succ_{-k}](i') = \varphi[\succ_k^*, \succ'_{-k}](i')$. Thus, $\varphi[\succ_k^*, \succ_{-k}](i)$ and $\varphi[\succ_k^*, \succ_{-k}](i')$ are different and both belong to $\{e, h\}$. .QED

Claim 4. If $\succ_k \in \langle e, h, g, ...\rangle$, $\varphi[\succ](k) = h$, and $\varphi[\succ'](k) = h$, then $\varphi[\succ_k^*, \succ_{-k}](i) = e$ and $\varphi[\succ_k^*, \succ_{-k}](i') = h$.

Proof of Claim 4. Let $\succ_k \in \langle e, h, g, ...\rangle$, $\varphi[\succ](k) = h$, and $\varphi[\succ'](k) = h$. Suppose that $\varphi[\succ_k^*, \succ_{-k}](i) \neq e$ for an indirect argument. Then, Claim 3 implies that $\varphi[\succ_k^*, \succ_{-k}](i) = h$ and $\varphi[\succ_k^*, \succ_{-k}](i') = e$. By Maskin monotonicity for i, $\varphi[\succ_k^*, \succ_{-k}] = \varphi[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}]$. By this equivalence and Claim 2, we have $\varphi[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}](k) = g$. By strategy-proofness, agent k gets at least g and not e at $[\succ_i^*, \succ_{-i}]$. Thus, $\varphi[\succ_i^*, \succ_{-i}](k) = h$. This contradicts Claim 1.

Thus, by Claim 3, $\varphi[\succ_k^*, \succ_{-k}](i) = e$ and $\varphi[\succ_k^*, \succ_{-k}](i') = h$. QED

Claim 5. If $\varphi[\succ](k) = h$ and $\varphi[\succ'](k) = h$ then $\varphi[\succ_{i'}^*, \succ'_{-i'}](i) \neq e$.

Proof of Claim 5. Let $\varphi[\succ'](k) = h$. First suppose that $\varphi[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}](i) = h$. We will show that this will lead to a contradiction: By Maskin monotonicity for i', $\varphi[\succ^*_i, \succ'_{-i}] = \varphi[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}]$, and in particular, $\varphi[\succ^*_i, \succ'_{-i}](i) = h$. By strategy-proofness for i, $\varphi[\succ'](i) \succeq_i h$, contradicting $\varphi[\succ'](i') = e$, $\varphi[\succ'](k) = h$, and thus, $\varphi[\succ'](i) \prec_i h$. A contradiction. Hence, $\varphi[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}](i) \neq h$.

Since $\succ^*_i$ replaces the ranking of h and e with respect to $\succ_i$, by Maskin monotonicity for i,

$$\varphi[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}] = \varphi[\succ^*_{i'}, \succ'_{-i'}]. \tag{2}$$

Let $\varphi[\succ](k) = h$. Symmetric argument with the above one implies that $\varphi[\succ^*_{\{i,i'\}}, \succ_{-\{i,i'\}}](i') \neq h$ and $\varphi[\succ^*_{\{i,i'\}}, \succ_{-\{i,i'\}}] = \varphi[\succ^*_i, \succ_{-i}]$. This and Claim 1 imply that $\varphi[\succ^*_{\{i,i'\}}, \succ_{-\{i,i'\}}](i) = h$.

Contrary to the claim, suppose that $\varphi[\succ^*_{i'}, \succ'_{-i'}](i) = e$. Then, $\varphi[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}](i) = e$ by Equation 2. Observe that at $[\succ^*_{\{i,i'\}}, \succ'_{-\{i,i'\}}]$, j envies i. By submitting $\succ_j$, agent j makes i better off (since $\varphi[\succ^*_{\{i,i'\}}, \succ_{-\{i,i'\}}](i) = h$), contradicting Lemma 4. Thus, we showed that $\varphi[\succ^*_{i'}, \succ'_{-i'}](i) \neq e$. QED

Claim 6. If $\varphi[\succ](k) = h$ and $\succ_i \in \langle e, h, g, ...\rangle$, then $\varphi[\succ'](k) = h$.

Proof of Claim 6. Let $\varphi[\succ](k) = h$ and $\succ_i \in \langle e, h, g, ...\rangle$. Since agent j envies i at $\succ$ and $\varphi[\succ](j) = g$, these and Corollary 1 imply that i gets at least g at $\succ'$. Hence, $\varphi[\succ'](i) \in \{h, g\}$ .However, by Lemma 4, j cannot continue envying i at $\succ'$. Hence, $\varphi[\succ'](i) = g$.

Let us first prove the claim under the additional assumption that $\succ_k \in \langle e, h, g, ...\rangle$. Since agent j envies k at $\succ$ and $\varphi[\succ](j) = g$, these and Corollary 1 imply that

$$\varphi[\succ'](k) \succeq_k g \tag{3}$$

Since $k \neq i'$ and $\varphi[\succ'](i') = e$, we have $\varphi[\succ'](k) \in \{h, g\}$. Suppose contrary to the claim that $\varphi[\succ'](k) \neq h = \varphi[\succ](k)$. Thus, by Lemma 4, j cannot envy k also at $\succ$; hence, $\varphi[\succ'](k) \neq e$. We know that $\varphi[\succ](i) = g$ and $i \neq k$. Thus, $\varphi[\succ'](k) \neq g$. Last three statements contradict Equation 3. We showed that j cannot change the allocation of k between $\succ$ and $\succ'$, and thus, $\varphi[\succ'](k) = \varphi[\succ](k) = h$.

Finally, the claim for general $\succ_k \in \langle e, h, ...\rangle$ follows by Maskin monotonicity of φ for k: for all $\succ_k \in \langle e, h, ...\rangle$ we have $\varphi[\succ'](k) = h$. QED

We are ready to complete the proof of the lemma as follows using the above claims:

Let's choose $k = \varphi^{-1}[\succ](h)$. Thus, $k \neq i, i', j$. By Maskin monotonicity, without loss of generality, we also choose $\succ_i \in \langle e, h, g, ...\rangle$, and $\succ_k \in \langle e, h, g, ...\rangle$ (recall that $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$). By Claim 6, $\varphi[\succ'](k) = h$. Thus, the hypotheses of Claims 1-5 hold. By Claim 1, $\varphi[\succ^*_{i'}, \succ'_{-i'}](i') = h$.

Hence, $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) = e$, $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) = g$, or $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) \prec_k g$. We will show that either of these cases leads to a contradiction, thus completing the indirect proof. First, we establish the following equations:

By Claim 2, we have $\varphi[\succ^*_k, \succ_{-k}](k) = g$. By Claim 4, $\varphi[\succ^*_k, \succ_{-k}](i') = h$. By Maskin monotonicity for i',

$$\varphi[\succ^*_k \succ^*_{i'}, \succ_{-\{k,i'\}}] = \varphi[\succ^*_k, \succ_{-k}]. \tag{4}$$

Hence,

$$\varphi[\succ^*_k, \succ^*_{i'}, \succ_{-\{k,i'\}}](k) = g. \tag{5}$$

By Maskin monotonicity for j, we have

$$\varphi[\succ^*_k \succ^*_{i'}, \succ'_{-\{k,i'\}}] = \varphi[\succ^*_k \succ^*_{i'}, \succ_{-\{k,i'\}}]. \tag{6}$$

Hence,

$$\varphi[\succ^*_k, \succ^*_{i'} \succ'_{-\{k,i'\}}](k) = g. \tag{7}$$

Case 1. $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) = e$: Agent k improves his allocation in Equation 7 by submitting $\succ_k$ instead of $\succ^*_k$, contradicting strategy-proofness.

Case 2. $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) = g$: By non-bossiness for k and Equation 7,

$$\varphi[\succ^*_k, \succ^*_{i'} \succ'_{-\{k,i'\}}] = \varphi[\succ^*_{i'}, \succ'_{-i'}]. \tag{8}$$

Equations 4, 6, and 8 imply that

$$\varphi[\succ^*_{i'}, \succ'_{-i'}] = \varphi[\succ^*_k, \succ_{-k}]. \tag{9}$$

By Claim 4, $\varphi[\succ^*_k, \succ_{-k}](i) = e$. By Equation 9, $\varphi[\succ^*_{i'}, \succ'_{-i'}](i) = e$. However, this contradicts Claim 5.

Case 3. $\varphi[\succ^*_{i'}, \succ'_{-i'}](k) \prec_k g$: When k submits $\succ^*_k$ instead of $\succ_k$, he improves his own allocation (since $\varphi[\succ^*_k, \succ^*_{i'}, \succ'_{-\{k,i'\}}](k) = g$ by Equation 7), a contradiction to strategy-proofness.

Thus, none of the three cases holds, implying that initial assumption $i' \neq i$ cannot be correct. **QED**

Lemma 8. *(**Existence and uniqueness of a broker* for each brokered* house**) Let $\sigma \in \overline{\mathcal{M}}$ and e be a brokered* house at σ. Then there exists an agent $k \in \overline{I_\sigma}$ who is the unique broker* of e at σ.*

Proof of Lemma 8. Let $\sigma \in \overline{\mathcal{M}}$ and e be a brokered* house at σ. We start with the following preparatory:

Claim 1. Let $h, h' \in \overline{H_\sigma} - \{e\}$ be such that $h \neq h'$, and let $\succ, \succ' \in \mathbf{P}[\sigma, e, h, h']$. Then $\varphi[\succ']^{-1}(h) = \varphi[\succ]^{-1}(h)$.

Proof of Claim 1. By Lemma 7, $\varphi[\succ']^{-1}(e) = \varphi[\succ]^{-1}(e)$. Let $i = \varphi[\succ]^{-1}(e)$. Also let $\succ^*$ and $\succ'^*$be monotonic extensions of $\succ$ and $\succ'$ respectively such that i ranks e first, all agents in $\overline{I_\sigma}$ rank e below all houses in $\overline{H_\sigma} - \{e\}$, and the relative ranking of all other houses at $\succ^*$, $\succ$ and $\succ'^*$, $\succ'$ are respectively the same. By Maskin monotonicity, $\varphi[\succ^{*\prime}] = \varphi[\succ']$ and $\varphi[\succ^*] = \varphi[\succ]$. Also $\succ^*, \succ'^* \in \mathbf{P}[\sigma \cup \{(i, e)\}, h, h']$. Thus, by Lemma 7, $\varphi[\succ^*]^{-1}(h) = \varphi[\succ^{*\prime}]^{-1}(h)$. Hence, $\varphi[\succ']^{-1}(h) = \varphi[\succ'^*]^{-1}(h) = \varphi[\succ^*]^{-1}(h) = \varphi[\succ]^{-1}(h)$. QED

Claim 2. Let $h, h' \in \overline{H_\sigma} - \{e\}$ be such that $h \neq h'$ and let $\succ \in \mathbf{P}[\sigma, e, h, h']$ and $\succ' \in \mathbf{P}[\sigma, e, h']$ such that $\varphi[\succ']^{-1}(e) \neq \varphi[\succ]^{-1}(e)$. Then $\varphi[\succ']^{-1}(h') = \varphi[\succ]^{-1}(h)$.

Proof of Claim 2. Let $k' = \varphi[\succ']^{-1}(h')$ and $\succ^* \in \mathbf{P}[\sigma, e, h', h]$ be such that the only difference between $\succ^*$ and $\succ$ is the relative ranking of house h'. Since by Claim 1 $\varphi[\succ^*]^{-1}(h') = \varphi[\succ']^{-1}(h') = k'$ and since we push down house h' in everybody's preferences except k' at $[\succ^*_{k'}, \succ_{-k'}]$, by Maskin monotonicity $\varphi[\succ^*_{k'}, \succ_{-k'}] = \varphi[\succ^*]$. In particular, $\varphi[\succ^*_{k'}, \succ_{-k'}](k') = h'$. By strategy-proofness for k', we have $\varphi[\succ](k') \in \{h, h'\}$. On the other hand, by Lemma 7, $\varphi[\succ^*]^{-1}(e) = \varphi[\succ']^{-1}(e)$. Since $\varphi[\succ^*_{k'}, \succ_{-k'}] = \varphi[\succ^*]$, we have $\varphi[\succ^*_{k'}, \succ_{-k'}]^{-1}(e) = \varphi[\succ']^{-1}(e)$.

We also have, $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e) = \varphi[\succ^*_{k'}, \succ_{-k'}]^{-1}(e)$. Thus, by non-bossiness, agent k' should change his own allocation between the two profiles $\succ$ and $[\succ^*_{k'}, \succ_{-k'}]$, implying that $\varphi[\succ](k') = h$. QED

Claim 3. Let $h, h' \in \overline{H_\sigma} - \{e\}$ be such that $h \neq h'$, $\succ \in \mathbf{P}[\sigma, e, h]$, and $\succ' \in \mathbf{P}[\sigma, e, h', h]$. Then, $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Proof of Claim 3. If $\varphi[\succ]^{-1}(e) \neq \varphi[\succ']^{-1}(e)$, then we are done by Claim 2. Therefore, assume that $\varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e)$. Because e is a brokered* house at σ, there exists some $h'' \in \overline{H_\sigma} - \{e\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, e, h'']$,

$$\varphi[\succ'']^{-1}(e) \neq \varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e).$$

By Lemma 7, $h'' \neq h$. By the same lemma, without loss of generality, we further assume that $\succ'' \in \mathbf{P}[\sigma, e, h'', h]$.

By Claim 2, $\varphi[\succ'']^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ'']^{-1}(h'') = \varphi[\succ']^{-1}(h')$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$. QED

Claim 4. Let $h \in \overline{H_\sigma} - \{e\}$ and $\succ, \succ' \in \mathbf{P}[\sigma, e, h]$. Then, $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h)$.

Proof of Claim 4. By Lemma 7, $\varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e)$. Because e is a brokered* house at σ, there exists some $h'' \in \overline{H_\sigma} - \{e\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, e, h'']$,

$$\varphi[\succ'']^{-1}(e) \neq \varphi[\succ]^{-1}(e) = \varphi[\succ']^{-1}(e).$$

By Lemma 7, $h'' \neq h$. Let $\succ^* \in \mathbf{P}[\sigma, e, h'', h]$. By Claim 3, $\varphi[\succ^*]^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ^*]^{-1}(h'') = \varphi[\succ']^{-1}(h)$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h)$. QED

We complete the proof of the lemma as follows: Let h and $h' \in \overline{H_\sigma} - \{e\}$, $\succ \in \mathbf{P}[\sigma, e, h]$, $\succ' \in \mathbf{P}[\sigma, e, h']$. Two cases are needed:

Case 1. $h = h'$: Then $\varphi[\succ']^{-1}(h) = \varphi[\succ]^{-1}(h)$ by Claim 4.

Case 2. $h \neq h'$: Then let $\succ^* \in \mathbf{P}[\sigma, e, h, h']$; by Claim 3 $\varphi[\succ']^{-1}(h) = \varphi[\succ^*]^{-1}(h')$ and by Claim 4 $\varphi[\succ^*]^{-1}(h) = \varphi[\succ]^{-1}(h)$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Thus, the agent $\varphi[\succ]^{-1}(h)$ is the unique broker* of e at σ. **QED**

Lemma 9. *Let $\sigma \in \overline{\mathcal{M}}$, $i \in \overline{I_\sigma}$, and $h \in \overline{H_\sigma}$. If $\varphi[\succ](i) = h$ for all $\succ \in \mathbf{P}^*[\sigma, h]$ then i owns* h at σ.*

Proof of Lemma 9. Let us start with two preparatory claims:

Claim 1. Let $\sigma \in \overline{\mathcal{M}}$, houses g and $h \in \overline{H_\sigma}$ be such that $g \neq h$, and agent $i \in \overline{I_\sigma}$ be such that $\varphi[\succ'](i) = h$ for all $\succ' \in \mathbf{P}[\sigma, g, h]$. Then $\varphi[\succ_i^*, \succ_{-i}](i) = g$ for all $\succ_i^* \in \langle g, ...\rangle$ and all $\succ_{-i} \in \mathbf{P}_{-i}[\sigma, h]$.

Proof of Claim 1. Let $\succ_{-i} \in \mathbf{P}_{-i}[\sigma, h]$. Take any $\succ_i \in \langle h, g, ...\rangle$. If $\varphi[\succ](i) = h$, then Pareto efficiency and strategy-proofness imply that $\varphi[\succ_i^*, \succ_{-i}](i) = g$ for all $\succ_i^* \in \langle g, h, ...\rangle$, and furthermore, by strategy-proofness, for all $\succ_i^* \in \langle g, ...\rangle$. It remains to consider the case $\varphi[\succ](i) \neq h$.

Take $\succ' \in \mathbf{P}[\sigma, h, g]$ such that $\succ'$ and $\succ$ coincide other than unmatched agents' ranking of house g. We have $\varphi[\succ'](i) = h$ by the hypothesis of the claim. Two cases are possible: $\varphi[\succ](i) = g$ and $\varphi[\succ](i) \neq g$. If $\varphi[\succ](i) = g$, then by strategy-proofness, $\varphi[\succ_i^*, \succ_{-i}](i) = g$ and we are done. Thus, in the remainder assume that there exists some agent $k = \varphi[\succ]^{-1}(g) \neq i$. By Maskin monotonicity, $\varphi[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}](i) = h$ and $\varphi[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}](k) = g$.

Let $\succ_i^* \in \langle g, h, ...\rangle$. By strategy-proofness, agent i gets at least h at $[\succ_i^*, \succ'_k, \succ_{-\{i,k\}}]$; and by Pareto efficiency, agent i gets g. Also recall that $\varphi[\succ](i) \prec_i g$ and $\varphi[\succ](k) = g$. Thus, $\varphi[\succ_i^*, \succ'_k$

$, \succ_{-\{i,k\}}](k) \neq h$ because otherwise agents i and k could jointly improve upon their $\varphi[\succ]$ allocation by submitting $[\succ_i^*, \succ_k']$ at $\succ$, contradicting group strategy-proofness. Thus, $g \succ_k' \varphi[\succ_i^*, \succ_k', \succ_{-\{i,k\}}](k)$, and furthermore, Maskin monotonicity implies $\varphi[\succ_i^*, \succ_k', \succ_{-\{i,k\}}] = \varphi[\succ_i^*, \succ_{-i}]$. In particular, $\varphi[\succ_i^*, \succ_{-i}](i) = g$. QED

Claim 2. Let $\sigma \in \overline{\mathcal{M}}$, houses g and $h \in \overline{H_\sigma}$ be such that $g \neq h$, $\succ \in \mathbf{P}[\sigma, h]$, and agent $i \in \overline{I_\sigma}$ such that $\varphi[\succ'](i) = h$ for all $\succ' \in \mathbf{P}[\sigma, g, h]$. If there is some $\succ' \in \mathbf{P}[\sigma, h, g]$ such that $\succ_k \in \langle h, g, ...\rangle$ for $k = \varphi[\succ']^{-1}(g)$, then $\varphi[\succ](i) = h$.

Proof of Claim 2. By way of contradiction, assume that i is the owner* of h at σ, that $\succ' \in \mathbf{P}[\sigma, h, g]$, and that $k = \varphi[\succ']^{-1}(g)$, but there is some $\succ \in \mathbf{P}[\sigma, h]$ such that $\succ_k \in \langle h, g, ...\rangle$ and $\varphi[\succ]^{-1}(h) \neq i$. By strategy-proofness, we can choose $\succ_i \in \langle h, g, ...\rangle$. Furthermore, we can choose $\succ$ such that $\succ$ and $\succ'$ differ only in preferences of a single agent $j \in \overline{I_\sigma}$ and in how house g is ranked by the agents.

Let $\succ^* \in \mathbf{P}[\sigma, h]$ be the unique profile, such that $\succ^*$ and $\succ$ differ only in the preferences of agent j, and $\succ^*$ and $\succ'$ differ only in how house g is ranked by the agents. Notice that $j \neq k$ as otherwise Maskin monotonicity would imply that i gets h at $\succ$. Thus, $\succ_k^* \in \langle h, g, ...\rangle$, and Maskin monotonicity implies that $\varphi[\succ^*](i) = h$.

Let h' be the house that j gets at $\succ$ and let $\succ''$ be the unique profile in $\mathbf{P}[\sigma, h, g]$ such that $\succ''$ and $\succ$ differ only in how house g is ranked by agents. By Maskin monotonicity, we may assume that $\succ_j'' \in \langle h, g, h', ...\rangle$.

By Claim 1 and strategy-proofness, $\varphi[\succ_j'', \succ_{-j}](i)$ equals either h or g. At the same time strategy-proofness implies that $\varphi[\succ_j'', \succ_{-j}](j)$ equals either g or h'. In either case, agent j prefers the allocation of agent i at $[\succ_j'', \succ_{-j}]$. If $\varphi[\succ_j'', \succ_{-j}](i) = g$, this would be a contradiction with Lemma 3, as j could improve the allocation of i by switching from $[\succ_j'', \succ_{-j}]$ to $[\succ_j^*, \succ_{-j}] = \succ^*$. Hence, $\varphi[\succ_j'', \succ_{-j}](i) = h$, and by non-bossiness $\varphi[\succ_j'', \succ_{-j}](j) = g$. However, $k \neq j$ gets g at $\succ'$ and by strategy-proofness j cannot get it at $[\succ_j'', \succ_{-j}']$. This is a contradiction because $[\succ_j'', \succ_{-j}] = [\succ_j'', \succ_{-j}']$. QED

We are ready to finish the proof of the lemma. Fix $\sigma \in \overline{\mathcal{M}}$. We proceed by way of contradiction. Let $i \in \overline{I_\sigma}$ be such that $\varphi[\succ'](i) = h$ for all $\succ' \in \mathbf{P}^*[\sigma, h]$. Let $\succ \in \mathbf{P}[\sigma, h]$ be such that $\varphi[\succ]^{-1}(h) = j \neq i$. For all unmatched houses $g \neq h$ at σ, define $\succ^g$ to be the unique profile in $\mathbf{P}[\sigma, h, g]$ that differs from $\succ$ only in how agents rank g.

Take a house $g_1 \neq h$ unmatched at σ, and let k_1 be the agent that gets g_1 at $\succ^{g_1}$. By Claim 2, agent i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_1 ranks g_1 second. Hence, by Maskin monotonicity i also gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_1 gets g_1.

Let $g_2 = \varphi[\succ](k_1)$ and let k_2 be the agent that gets g_2 at $\succ^{g_2}$. Because i does not get h at $\succ$, the previous paragraph yields $g_2 \neq g_1$ and $k_2 \neq k_1$. As in the previous paragraph, Claim 2 and Maskin monotonicity imply that i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_2 gets g_2 or ranks g_2 second.

Furthermore, we will show that i gets h at any profile $\succ' \in \mathbf{P}[\sigma, h]$ at which k_2 ranks g_1 second. Indeed, suppose $\succ'_{k_2} \in \langle h, g_1, ... \rangle$ and i does not get h at $\succ'$. Let $\succ''_i \in \langle h, g_1, ... \rangle$. By Claim 1 and strategy-proofness, agent i gets g_1 at $[\succ''_i, \succ'_{-i}]$. By the previous paragraph and strategy-proofness, k_2 does not get h at $[\succ''_i, \succ'_{-i}]$, and thus k_2 envies i at $[\succ''_i, \succ'_{-i}]$. However, by the previous paragraph k_2 can improve the outcome of agent i, contrary to Lemma 4. Thus, i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_2 ranks g_1 second.

Let g_3 be the house that k_2 gets at $\succ$ and let k_3 be the agent that gets g_3 at $\succ^{g_3}$. As above, we can show that i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_3 ranks g_3 or g_2 or g_1 second.

Since the number of agents is finite, by repeating the procedure we arrive at an agent k_n who ranks one of the houses $g_1, ..., g_n$ second at $\succ$. That means that i gets h at $\succ$, a contradiction that concludes the proof. **QED**

Lemmas 8 and 9 give us the key result of this subsection:

Corollary 2. ***(Houses are either brokered* or owned*)*** *For any $\sigma \in \overline{\mathcal{M}}$, any house $h \in \overline{H_\sigma}$ is either owned* or brokered* at σ.*

C.2 The Starred Control Right Structure Satisfies R1-R6

Before proving R1-R6 let us state and prove one more auxiliary result.

Lemma 10. ***(Relationship between brokerage* and ownership*).*** *Let $\sigma \in \overline{\mathcal{M}}$, agent k be a broker* of house e at σ, and $\succ'' \in \mathbf{P}^*[\sigma, e]$. Then agent $\varphi[\succ'']^{-1}(e)$ is the owner* of house $\varphi[\succ''](k)$ at σ.*

Proof of Lemma 10. Let $\succ'' \in \mathbf{P}^*[\sigma, e]$ and $h = \varphi[\succ''](k)$. Because k is a broker* at σ, Lemma 8 implies that house h is agent k's second choice. Since $\succ'' \in \mathbf{P}^*[\sigma, e]$, house h is the second choice of all agents in $\overline{I_\sigma}$ at $\succ''$, and thus,

$$\succ'' \in \mathbf{P}[\sigma, e, h].$$

There exists an agent $i \in (\overline{I_\sigma}) - \{k\}$ such that $\varphi[\succ'']^{-1}(e) = i$. By Lemma 7, for all $\succ \in \mathbf{P}[\sigma, e, h]$, agent i gets e at $\succ$. We are to show that i is the owner* of h at σ.

Claim 1. If $\succ \in \mathbf{P}[\sigma, e, h]$, then $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$.

Proof of Claim 1. The first claim follows from Lemma 7, and the second from Lemma 8. QED

Claim 2. $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$.

*Proof of Claim 2.*Let preference profile $\succ$ be such that $\succ_{i'}=\succ''_{i'}$ for all $i' \in \{k,i\} \cup I_\sigma$ and all houses in $\overline{H_\sigma}$ are ranked above the houses in H_σ by $i' \in \overline{I_\sigma}$. By Claim 1 and Maskin monotonicity, $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$. QED.

Claim 3. $\varphi[\succ^*_i, \succ_{-i}](i) = h$.

Proof of Claim 3. Let $\succ^*_i \in \langle h, e, ...\rangle$. By strategy-proofness of φ, since $\varphi[\succ](i) = e$, agent i gets at least e at $[\succ^*_i, \succ_{-i}]$, and since all other agents in $\overline{I_\sigma}$ prefer e over h, Pareto efficiency of φ implies that $\varphi[\succ^*_i, \succ_{-i}](i) = h$.

Claim 4. $\varphi[\succ^*_k, \succ_{-k}] = \varphi[\succ]$.

Proof of Claim 4. Let $\succ^*_k \in \langle h, e, ...\rangle$. Since $\varphi[\succ](k) = h$, profile $[\succ^*_k, \succ_{-k}]$ is a monotonic transformation of $\succ$ and by Maskin monotonicity of φ, we have $\varphi[\succ^*_k, \succ_{-k}] = \varphi[\succ]$. *Claim 5.* $\varphi[\succ^*_{\{i,k\}}$

$, \succ_{-\{i,k\}}](i) = h$.

Proof of Claim 5. By Claim 4, $\varphi[\succ^*_k, \succ_{-k}](i) = \varphi[\succ](i) = e$, and, by strategy-proofness of φ, i gets at least e at $[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}]$. Thus, if i does not get h at $[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}]$ then one of the following two cases would have to obtain.

Case 1. An agent $j \notin \{i, k\}$ gets h at $[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}]$: Then i gets e, and k gets some house worse than e. But then jointly i and k can report $\succ_{\{i,k\}}$ instead of $\succ^*_{\{i,k\}}$ and they would jointly improve at $\succ^*_{\{i,k\}}$, i.e., $\varphi[\succ](i) = e = \varphi[\succ^*_{i,k}, \succ_{-i,k}](i)$ and $\varphi[\succ](k) = h \succ^*_k \varphi[\succ^*_{i,k}, \succ_{-i,k}](k)$, contradicting φ is group strategy-proof.

Case 2. Agent k gets h at $[\succ^*_{i,k}, \succ_{-i,k}]$: By strategy-proofness of φ, agent k should at least get h at $[\succ^*_i, \succ_{-i}]$. But we know by Step 2 that $\varphi[\succ^*_i, \succ_{-i}](i) = h$, thus we should have $\varphi[\succ^*_i, \succ_{-i}](k) = e$. Then by Maskin monotonicity of φ, we have $\varphi[\succ^*_{i,k}, \succ_{-i,k}](i) = \varphi[\succ^*_i, \succ_{-i}](i) = h$ where the last equality follows by Step 2. A contradiction that proves the claim. QED

Claim 6. If $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}](i) = h$, then $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}](k) \neq e$.

Proof of Claim 6. For an indirect argument, suppose that $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}](i) = h$ and $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}$ $](k) = e$. Then, $\varphi[\succ^*_i, \succ_{-i}](k) = e$ by strategy-proofness of φ. Since e is a brokered* house at σ, there exist some house $g \notin \{e, h\}$ and some preference profile $\succ' \in \mathbf{P}[\sigma, e, g]$ such that $\varphi[\succ']^{-1}(e) = j$ for some agent $j \notin \{i, k\}$. By Lemma 7, we may assume that each agent $i' \in \overline{I_\sigma}$ ranks houses other than g and h in the same way at $\succ'_{i'}$ and $\succ_{i'}$ and that $\succ'_{i'} \in \langle e, g, h, ...\rangle$. Since k is the broker* of e at σ, we have $\varphi[\succ'](k) = g$. By Maskin monotonicity,

$$\varphi[\succ'] = \varphi[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}].$$

Now i gets a house weakly worse than h at $[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}]$. However, if i and k manipulated and submitted $\succ^*_{\{i,k\}}$ instead of $\succ'_{\{i,k\}}$, they would get h and e respectively at $[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}]$. Both agents weakly improve, while k strictly improves. This contradicts the fact that φ is group strategy-proof. QED

Now, Claims 5 and 6 imply that $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}](i) = h$ and $\varphi[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}](k) \neq e$. By Maskin monotonicity, we can drop the ranking of e in $\succ^*_i$ and $\succ^*_k$, and yet, the outcome of φ will not change. Recall that $\succ_{-\{i,k\}}$ was an arbitrary profile in which all houses in $\overline{H_\sigma}$ are ranked above the houses in H_σ by $i' \in \overline{I_\sigma} - \{i,k\}$. Thus, i gets h at all profiles of $\mathbf{P}[\sigma, h]$. **QED**

The following six lemmas show that the starred control right structure satisfies R1-R6 (respectively).

Lemma 11. ***(R1; Uniqueness of a brokered* house)****. Let $\sigma \in \overline{\mathcal{M}}$. If e is a brokered* house at σ, then no other house is a brokered* house at σ (and all other unmatched houses are owned* houses).*

Proof of Lemma 11. Let e be a brokered* house at σ. By Lemma 8, there is a broker* of e at σ, let us denote him as k. Consider a house $h \in \overline{I_\sigma} - \{e\}$. By Lemma 7, there is an agent i who gets e at all profiles in $\mathbf{P}[\sigma, e, h]$. By Lemma 9, i is the owner* of h. Thus h is not a brokered* house at σ. **QED**

Lemma 12. ***(R2; Last unmatched agent is an owner)****. Let $\sigma \in \overline{\mathcal{M}}$, such that there exists a unique agent i unmatched at σ. Then i owns* all unmatched houses at $\sigma \in \overline{I_\sigma}$.*

Proof of Lemma 12. Let $\succ \in \mathbf{P}[\sigma, h]$ for $h \in \overline{H_\sigma}$. By Pareto efficiency of φ, $\varphi[\succ](i) = h$, implying that i owns* h at σ. **QED**

Lemma 13. ***(R3; Broker* does not own*)****. Let $\sigma \in \overline{\mathcal{M}}$. If agent k is the broker* of house e at σ, then he cannot own* any houses at σ.*

Proof of Lemma 13. Suppose that k owns* a house $h \neq e$ at σ. By Lemma 7, there exists some agent $i \neq k$ who gets e at all profiles in $\mathbf{P}[\sigma, e, h]$. Thus, i gets h at all $\succ \in \mathbf{P}^*[\sigma, h]$, contradicting that k owns* h. **QED**

Lemma 14. ***(R4; Persistence of ownership*)****. Let i own* h at some $\sigma \in \overline{\mathcal{M}}$. If $\sigma' \supsetneq \sigma$, and i and h are unmatched at σ', then i owns* h at σ'.*

Proof of Lemma 14. Imagine to the contrary that i gets h at all $\succ \in \mathbf{P}[\sigma, h]$, but there is some $\succ' \in \mathbf{P}[\sigma', h]$ such that some agent $j \in I_{\sigma'} - I_\sigma$, such that $j \neq i$, gets h at $\succ'$. Take $\succ \in \mathbf{P}[\sigma, h]$ such that

- for each agent $k \notin I_{\sigma'} - I_\sigma$, $\succ_k = \succ'_k$, and
- each agent $k \in I_{\sigma'} - I_\sigma$ ranks $\sigma'(k)$ as his second choice (just behind h) in $\succ_k$.

Each $k \in I_{\sigma'} - I_\sigma$ is indifferent between $\succ'$ and $\succ$ because:

- at $\succ'$ agent k gets $\sigma'(k)$ by Lemma 6,
- at $\succ$ agent k gets $\sigma'(k)$ by Pareto efficiency of φ and the fact that $\varphi[\succ](i) = h$.

The only difference between the profiles $\succ'$ and $\succ$ are the preferences of the agents in $I_{\sigma'} - I_\sigma$. Thus, agents $I_{\sigma'} - I_\sigma$ are indifferent between $\succ$ to $\succ'$ while agent j is strictly better off at $\succ'$. This contradicts the fact that φ is group strategy-proof. **QED**

Lemma 15. ***(R5; Limited persistence of brokerage*)*** *Let $\sigma, \sigma' \in \overline{\mathcal{M}}$ be such that $\sigma' \supsetneq \sigma$. Suppose that agent k is the broker* of house e at σ, agent i is the owner* of house h at σ, and agent $i' \neq i$ is the owner* of house h' at σ. If k, i, i', e, h, h' are unmatched at σ', then k brokers* e at σ'.*

Proof of Lemma 15. First, notice that i gets e at all $\succ \in \mathbf{P}[\sigma, e, h]$ and i' gets e at all $\succ \in \mathbf{P}[\sigma, e, h']$, and k gets h and h', respectively by Lemma 10. Take $\succ^h \in \mathbf{P}[\sigma, e, h]$ and $\succ^{h'} \in \mathbf{P}[\sigma, e, h']$ such that each agent $j \in I_{\sigma'} - I_\sigma$ has $\sigma'(j)$ as his third choice and each agent $j \in I - I_{\sigma'}$ ranks each house unmatched at σ' above all houses matched at σ' at both preference profiles. Let profile $\succ'^h$ be obtained from $\succ^h$ by moving $\sigma'(j)$ for all $j \in I_{\sigma'} - I_\sigma$ up to be the first choice of j. Let $\succ'^{h'}$be obtained analogously from $\succ^{h'}$. By Maskin monotonicity, $\varphi[\succ'^h]^{-1}(e) = i \neq i' = \varphi[\succ'^{h'}]^{-1}(e)$. Since $\succ'^h$ and $\succ'^{h'} \in \mathbf{P}^*[\sigma', e]$, house e is a brokered* house at σ'.

For an indirect argument for the second part of the proof, suppose that k is not the broker* of e at σ'. Then, by Lemma 8 there exists some other agent $k' \neq k$ who brokers* e at σ'.

Let $\succ' \in \mathbf{P}[\sigma', e, h]$ be arbitrary and $\succ \in \mathbf{P}[\sigma, e, h]$ be such that each agent j in $I_{\sigma'} - I_\sigma$ lists $\sigma'(j)$ as his third choice at $\succ$, each agent in $I - I_{\sigma'}$ lists houses in $H_{\sigma'}$ lower than houses in $H_{\sigma'} - H_\sigma$ at $\succ$, and rest of the relative rankings of the houses are the same between $\succ$ and $\succ'$. Since k brokers* e at σ and i owns* h at σ, by Lemma 10 $\varphi[\succ](k) = g$ and $\varphi[\succ](i) = e$. Then, by Pareto efficiency, $\varphi[\succ'](j) = \sigma'(j)$ for all $j \in I_{\sigma'} - I_\sigma$, and thus, by Maskin monotonicity, $\varphi[\succ'] = \varphi[\succ]$. Now, $\varphi[\succ'](k) = h$, however, this contradicts the fact that agent $k' \neq k$ brokers* e at σ' and thus, $\varphi[\succ'](k') = h$. Therefore, k brokers* e at σ', as well. **QED**

Lemma 16. ***(R6; Broker*-to-heir* transition)*** *Let $\sigma \in \overline{\mathcal{M}}$, $k, j, i \in \overline{I_\sigma}$, and $e, g, h \in \overline{H_\sigma}$ be such that $k \neq j$ brokers* $e \neq g$ at σ, i is the only owner* who is unmatched at both σ and $\sigma' = \sigma \cup \{(j, g)\}$ and owns* h both at σ and σ'. Further suppose that k no longer brokers e at σ'. Then i owns* e at σ' and k owns* h at $\sigma' \cup \{(i, e)\}$.*

Proof of Lemma 16. By Lemmata 9 and 10 and Maskin monotonicity, for all profiles $\succ \in \mathbf{P}[\sigma]$ such that $\succ_i \in \langle e, ...\rangle$, $\succ_k \in \langle e, h, ...\rangle$ at σ, we have $\varphi[\succ](i) = e$ and $\varphi[\succ](k) = h$. Since $\mathbf{P}[\sigma'] \subset \mathbf{P}[\sigma]$, Corollary 2 implies that either i owns* e at σ' or k brokers* e at σ'. The latter is not true by an assumption made in the lemma, hence i owns* e at σ'. Let $\succ' \in \mathbf{P}[\sigma' \cup \{(i, e)\}, h]$. Fix a profile $\succ$ as described above with the further restriction that $\succ \in \mathbf{P}[\sigma', e]$ and relative ranking of all houses except h and e coincides with that of $\succ'$. Thus, $\succ'$ is a monotonic transformation of $\succ$, implying that $\varphi[\succ'] = \varphi[\succ]$, and in particular, $\varphi[\succ'](k) = \varphi[\succ](k) = h$. Thus, k owns* h at $\sigma' \cup \{(i, e)\}$. **QED**

C.3 The TCBO Mechanism Defined by the Starred Control Right Structure Equals φ

We showed above that the starred control right structure (c, b) is well-defined and consistent (satisfies R1-R6). We will now close the prove of Theorem 2 by showing that the resulting TCBO mechanism, $\psi^{c,b}$, maps preferences to outcomes in the same way as φ does. We will proceed by induction on rounds of $\psi^{c,b}$.

Fix $\succ \in \mathbf{P}$. We will show that $\varphi[\succ] = \psi^{c,b}[\succ]$. Let I^r be the set of agents removed in round r of $\psi^{c,b}$. For each agent $i \in I^r$, there is a unique house that points to him and is removed in the same cycle as i; let us denote this house h_i. Let us construct the following preference profile $\succ^*$ by modifying $\succ$.

- If $\psi^{c,b}[\succ](i) = h_i$, then $\succ_i^* = \succ_i$.
- If $\psi^{c,b}[\succ](i) \neq h_i$ and if no brokered house was removed in the same cycle as i or the brokered house was assigned to i, then we construct $\succ_i^*$ from $\succ_i$ by moving h_i just after $\psi^{c,b}[\succ](i)$ (we do not change the ranking of other houses).
- If i is removed as owner and a brokered house $e^r \neq \psi^{c,b}[\succ](i)$ was removed in the same cycle as i, then we construct $\succ_i^*$ from $\succ_i$ by moving e^r just after $\psi^{c,b}[\succ](i)$ and moving h_i just after e^r.
- If a broker k^r was removed in the cycle

$$h_{i^1} \to i^1 \to h_{i^2} \to i^2 \to ...h_{i^n} \to i^n \to e^r \to k^r \to h_{i^1},$$

then we construct $\succ^*_{k^r}$ from $\succ_{k^r}$ by moving h_{i^n} just below h_{i^1}.

Observe that $\psi^{c,b}[\succ^*] = \psi^{c,b}[\succ]$. Moreover, since

$$\left\{ h \in H : h \succeq_i \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} = \left\{ h \in H : h \succeq^*_i \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} \qquad \forall i \in I, \tag{10}$$

$\succ^*$ is a monotonic transformation of $\succ$ at $\psi^{c,b}$ and $\succ$ is a monotonic transformation of $\succ^*$ at $\psi^{c,b}$.

We will next prove that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r} I^s = I_{\sigma^r}, \qquad \forall r = 0, 1, 2, ... \tag{11}$$

by induction over r. The claim is trivially true for $r = 0$. Fix round $r \geq 1$ and let σ^{r-1} be the matching fixed before round r (in particular, $\sigma^0 = \varnothing$). For the inductive step, assume that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r-1} I^s = I_{\sigma^{r-1}} \tag{12}$$

We will prove that the same expression holds for agents in I^r using the following three claims (Editorial comment: fix numbering of claims; there is no claim 1 anymore).

Claim 2. $\varphi[\succ^*](i) \succeq^*_i h_i$ for all owners $i \in I^r$.

Proof of Claim 2. Let $\succ' \in \mathbf{P}[\sigma^{r-1}, h_i]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}} - \{h_i\}$ in $\succ'_j$ is the same as in $\succ^*_j$ for all $j \in (I - I_{\sigma^{r-1}}) - \{i\}$, and let $\succ'' \in \mathbf{P}[\sigma^{r-1}]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ in $\succ''_j$ is the same as in $\succ^*_j$ for all $j \in (I - I_{\sigma^{r-1}}) - \{i\}$.

If $i' \in I_{\sigma^{r-1}}$ then

$$\varphi[\succ'](i') = \varphi[\succ''](i') = \sigma^{r-1}(i') = \psi^{c,b}[\succ^*](i') = \varphi[\succ^*](i'),$$

by construction of $\mathbf{P}[\sigma^{r-1}, h_i], \mathbf{P}[\sigma^{r-1}]$, and σ^{r-1}, and by the inductive assumption. Since i owns h_i at σ^{r-1}, he owns* it by construction of (c, b), and thus,

$$\varphi[\succ'](i) = h_i. \tag{13}$$

Thus, no agent $j \in (I - I_{\sigma^{r-1}}) - \{i\}$ gets a house in $\{h_i\} \cup H_{\sigma^{r-1}}$ at $\varphi[\succ']$.

By Maskin monotonicity,

$$\begin{aligned} \varphi[\succ^*] &= \varphi[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ^*_{I_{\sigma^{r-1}} \cup \{i\}}] \\ &= \varphi[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ'_{I_{\sigma^{r-1}}}, \succ^*_i], \end{aligned} \tag{14}$$

and

$$\varphi[\succ'] = \varphi[\succ''_{(I-I_{\sigma^{r-1}})-\{i\}}, \succ'_{I_{\sigma^{r-1}}\cup\{i\}}]. \tag{15}$$

By Equation 14, strategy-proofness of φ, and Equations 15 and 13, we have

$$\varphi[\succ^*](i) = \varphi[\succ''_{(I-I_{\sigma^{r-1}})-\{i\}}, \succ'_{I_{\sigma^{r-1}}}, \succ^*_i](i) \;\; \succeq^*_i \;\; \varphi[\succ''_{(I-I_{\sigma^{r-1}})-\{i\}}, \succ'_{I_{\sigma^{r-1}}\cup\{i\}}](i) = \varphi[\succ'](i) = h_i.$$

QED

Claim 3. If $i \in I^r$ and no brokered house was removed in the cycle of i, then $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$.

Proof of Claim 3. The inductive assumption implies that all houses better than $\psi^{c,b}[\succ^*](i)$ are already given to other agents, hence

$$\psi^{c,b}[\succ^*](i) \succeq^*_i \varphi[\succ^*](i).$$

For an indirect argument, suppose $\varphi[\succ^*](i) \neq \psi^{c,b}[\succ^*](i)$. Then, Claim 2 and the construction of $\succ^*$ imply that

$$\varphi[\succ^*](i) = h_i.$$

Let

$$h_i \to i \to h_{i^2} \to i^2 \to ... \to h_{i^n} \to i^n \to h_i$$

be the cycle in which i is removed under $\psi^{c,b}[\succ^*]$. From

$$\varphi[\succ^*](i) = h_i = \psi^{c,b}[\succ^*](i^n),$$

we conclude that $\varphi[\succ^*](i^n) \neq \psi^{c,b}[\succ^*](i^n)$, and Claim 2 and the construction of $\succ^*$ imply that

$$\varphi[\succ^*](i^n) = h_{i^n} = \psi^{c,b}[\succ^*](i^{n-1}).$$

As we continue iteratively, we obtain that

$$\varphi[\succ^*](j) = h_j$$

for all $j \in \{i, i^2, ..., i^n\}$. Hence, the matching obtained by assigning $\psi^{c,b}[\succ^*](j)$ to each agent $j \in \{i, i^2, ..., i^n\}$ and $\varphi[\succ^*](j)$ to each agent $j \in I - \{i, i^2, ..., i^n\}$ Pareto dominates $\varphi[\succ^*]$ at $\succ^*$, contradicting that $\varphi[\succ^*]$ is Pareto efficient. QED

Claim 4. If $i \in I^r$ and a brokered house was removed in the cycle of i, then $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$.

Proof of Claim 4. Let e be the brokered house at σ^{r-1}

$$h_{i^1} \to i^1 \to h_{i^2} \to ... \to i^n \to e \to k \to h_{i^1}$$

Let

$$h_{i^{n+1}} \equiv e \qquad \text{and} \qquad i^0 \equiv k.$$

For all $i^\ell \in \{i^1, ..., i^n\}$, by the inductive assumption, all houses better than $h_{i^{\ell+1}}$ are already given to other agents, hence Claim 2 implies that

$$\varphi[\succ^*](i^\ell) \in \{h_{i^{\ell+1}}, e, h_{i^\ell}\}, \tag{16}$$

where $h^*_{i^{\ell+1}} \succeq^*_{i^\ell} e \succ_{i^\ell} h_{i^\ell}$. We prove the claim in two steps:

- First, we show that
$$\varphi[\succ^*](i^n) = e = \psi^{c,b}[\succ^*](i^n).$$
Suppose on the contrary that $\varphi[\succ^*](i^n) \neq e$. Then, $\varphi[\succ^*](i^n) = h_{i^n}$ by Equation 16. By iteration of the same argument for $\ell = n-1, n-2, ..., 1$, we have
$$\varphi[\succ^*](i^\ell) \in \{e, h_{i^\ell}\}. \tag{17}$$
Recall that by construction $e \succ^*_{i^\ell} h_{i^\ell}$. Let $\succ' \in \mathbf{P}[\sigma^{r-1}]$ be such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ at $\succ'_j$ is the same as at $\succ^*_j$ for all $j \in I - \{k, i^1\}$ and $\succ'_k, \succ'_{i^1} \in \langle e, h_{i^1}, ...\rangle$. Then by Lemma 10, $\varphi[\succ'](i^1) = e$ and $\varphi[\succ'](k) = h_{i^1}$. Thus, by Maskin monotonicity, $\varphi[\succ'] = \varphi[\succ^*]$, implying that $\varphi[\succ^*](i^1) = e$ and $\varphi[\succ'](k) = h_{i^1}$. Moreover, Equation 17 implies that $\varphi[\succ^*](i^\ell) = h_{i^\ell} \quad \forall \ell \in \{1, ..., n-1\}$. However, the matching that assigns each agent in $i^\ell \in \{i^1, ..., i^n\}$ the house $h_{i^{\ell+1}}$ and every other agent j the house $\varphi[\succ^*](j)$ Pareto dominates $\varphi[\succ^*]$, contradicting Pareto efficiency of φ.

- Next, we show that
$$\varphi[\succ^*](i^\ell) = h_{i^{\ell+1}} = \psi^{c,b}[\succ^*](i^\ell) \qquad \forall\, \ell \in \{0, ..., n-1\}.$$
On the contrary, suppose there exists some $\ell \in \{0, ..., n-1\}$ such that $\varphi[\succ^*](i^\ell) \neq h_{i^{\ell+1}}$. Thus, by Equation 16 and the fact that $\varphi[\succ^*](i^n) = e$, we have $\varphi[\succ^*](i^\ell) = h_{i^\ell}$. By iteration of this argument for all $m = \ell-1, \ell-2, ..., 1$, $\varphi[\succ^*](i^m) = h_{i^m}$. Thus, $\varphi[\succ^*](k) \neq h_{i^1}$. Let $\succ' \in \mathbf{P}[\sigma^{r-1}]$ be such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ at $\succ'_j$ is the same as at $\succ^*_j$ for all $j \in I - \{k, i^n\}$, and $\succ'_k, \succ'_{i^n} \in \langle e, h_{i^n}, ...\rangle$. Then by Lemma 10, $\varphi[\succ'](i^n) = e$ and $\varphi[\succ'](k) = h_{i^n}$. By $\varphi[\succ^*](k) \neq h_{i^1}$ and then Maskin monotonicity, $\varphi[\succ^*] = \varphi[\succ']$, and in particular, $\varphi[\succ^*](k) = h_{i^n}$. Thus, by Equation 16, $\varphi[\succ'](i^{n-1}) = h_{i^{n-1}}$. By iteration of the same argument for all $\ell = n-2, n-3, ..., \ell+1$,$\varphi[\succ'](i^m) = h_{i^m}$. On the other hand, the matching which assigns each agent $i^\ell \in \{i^1, ..., i^{n-1}\}$ house $h_{i^{\ell+1}}$, agent k house h_{i^1} and all other agents their houses at $\varphi[\succ^*]$ Pareto dominates $\varphi[\succ^*]$, contradicting Pareto efficiency of φ. QED

Let σ^r be the matching fixed after Round r. By the inductive assumption, and by Claims 3 and 4, $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$ for all $i \in I_{\sigma^r}$. This completes the induction, and the proof of Statement in (11) (i.e., Equation 11).

The theorem follows from

$$\psi^{c,b}[\succ] = \psi^{c,b}[\succ^*], \qquad \psi^{c,b}[\succ^*] = \varphi[\succ^*], \qquad \text{and} \qquad \varphi[\succ^*] = \varphi[\succ].$$

The first of these observations is straightforward through the construction of $\succ^*$. The second one follows from Equation 11. The third one follows from Maskin monotonicity of φ, because $\psi^{c,b}[\succ^*] = \varphi[\succ^*]$ and Equation 10 together imply that

$$\{h \in H : h \succeq_i \varphi[\succ^*](i)\} = \{h \in H : h \succeq_i^* \varphi[\succ^*](i)\} \text{ for all } i \in I.$$

QED

D Appendix: Proof of Theorem 5

The argument for Pareto efficiency of TCBO remains the same as in the TCBO example of Section 3.2. As before group strategy-proofness is equivalent to individual strategy-proofness and non-bossiness.

Lemma 1. *In the environment with outside options, a mechanism is group strategy-proof if and only if it is individually strategy-proof and non-bossy.*

The proof follows word-by-word the proof of Lemma 1 in Pápai [2000]. **QED**

Our arguments for individual strategy-proofness and non-bossiness go through with two modifications. First, when in the proof of Theorem 1 we assume that an agent is matched with a house, we should now substitute "a house or the agent's outside option." If the agent is matched in a cycle of length above 1, we can then conclude that the agent is indeed matched with a house. Second, in some steps of the proof we consider separately the case when a broker is matched with his outside option. We handle these cases below. This allows us to assume this case away in the relevant parts of the original proof.

Consider the proof of individual strategy-proofness. In Case 1: $s \le s'$, let i be a broker of house e and under $\succ_i$ leaves with his outside option in round s. Since the same houses are matched under $\succ_i$ and $\succ_i'$, under $\succ_i'$ the best the broker can do is to leave either with his outside option, or – if he prefers the brokered house e to his outside option – to leave with the brokered house e. We need to prove that the latter cannot happen. By Lemma 3, in round s of TCBO under $\succ_i'$, agent i is a broker of e and there is an owner j whose first preference is e. For i to be matched with e, he would need to lose brokerage right but by R5-R6 if this happens then j becomes the owner of e, and is then matched with it, ending the argument for Case 1. In Case 2: $s > s'$, if i be a broker of house e matched with his outside option under $\succ_i'$, then submitting this preference profile cannot be better than submitting the true profile $\succ_i$ as under any profile agent i is matched at least with his outside option.

Consider the proof of non-bossiness. We run the same induction as in the proof without outside options. In the initial step of the induction, consider the additional case when i_* is a broker and is matched with his outside option at time s under $\succ$. By assumption i_* is matched with his outside option under $\succ'$ and the inductive hypothesis is true. In the inductive step, consider the additional case in which i^1 is a broker and is matched with his outside option at time $r > s$ under $\succ$ (handling this case separately allows us to assume this case away in all claims of the inductive step). By the inductive assumption, there is r^* such that $\sigma^{r-1}[\succ] \subseteq \sigma^{r^*}[\succ']$. At $\sigma^{r-1}[\succ]$, i^1 brokers a house h and all houses other than h that i^1 prefers to his outside option are matched. Since i^1 gets at least his outside option, he either gets his outside option (and the inductive step is true) or he gets h. In the latter case, as in the proof of individual strategy-proofness, at $\sigma^{r-1}[\succ]$, there is an owner j at whose top preference is h. He remains unmatched as long as h is unmatched. Since for i^1 to obtain h he would need to lose his brokerage right, conditions R5-R6 imply that j would get ownership over h, and would match with h. Hence i^1 cannot be matched with h and is matched with his outside option.

To prove that any group strategy-proof and efficient mechanism is TCBO we follow the same steps as in the proof of Theorem 2 with one important modification. For $\sigma \in \overline{\mathcal{M}}$, $n \geq 0$ and $h^1, h^2, ..., h^n \in \overline{H_\sigma}$, and $i \in I$, we re-define $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ to be the set of preferences $\succ_i$ of agent i such that

- if $i \in I_\sigma$, then
$$\sigma(i) \succ_i g \text{ for all } g \in H - \{\sigma(i)\},$$
- if $i \in \overline{I_\sigma}$, then
$$h^1 \succeq_i h^2 \succeq ... \succeq_i h^n \succ_i y_i \succ g \text{ for all } g \in H_\sigma.$$

In particular, the definitions of ownership* and brokerage* are repeated word-by-word, but the meaning of $\mathbf{P}_i[\sigma, h^1, ..., h^n]$ is changed as above. With this modification, the proof goes through. **QED**

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